STABLE REPRESENTATION THEORY OF CATEGORIES AND APPLICATIONS TO FAMILIES OF (BI)MODULES OVER SYMMETRIC GROUPS

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STABLE REPRESENTATION THEORY OF CATEGORIES AND APPLICATIONS TO FAMILIES OF (BI)MODULES OVER SYMMETRIC GROUPS

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This work deals with the stable representation theory of categories related to various families of symmetric groups. In particular, we study the categories FB, FI, and introduce the new category PD. The notion of representation stability is recast in the setting of FB-bimodules.

The first part of the present work is an application of theory of FI-modules to a family of groups $\Gamma_{n,s}$ arising in the study of free group automorphisms. We observe that the cohomology of these groups determines an FI-module $H^i(\Gamma_{n,\bullet})$ which we show is finitely generated of stability degree n and weight i. It follows that the sequence $\{H^i(\Gamma_{n,s})\}_s$ is representation stable in the range $s \geq i+n$, an improvement on the previously known stable range. Another consequence of this finitely generated FI-module structure is the existence of character polynomials which determine the stable characters of $H^i(\Gamma_{n,s})$. In particular, this implies that the dimension of $H^i(\Gamma_{n,s})$ is given by a single polynomial in s for $s \geq i+n$. We compute explicit examples of such character polynomials to demonstrate this phenomenon.

Next we provide an algorithm that computes certain structural coefficients $c_{\lambda\mu}$ related to the n-th tensor power of the free associative algebra on a vector space $\mathcal{T}(V)^{\otimes n}$. By extending the known range of computation by a factor of over 750 we reveal striking patterns that motivate our recasting of representation stability to families of bimodules.

Finally, we develop the theory of PD-modules. Our main result is that finitely generated PD-modules give rise to representation stable families of bimodules over symmetric groups. We provide two main examples of this framework. First we show

that the coefficients $c_{\lambda\mu}$ determine a finitely generated PD-module and thus provide a first example of this new representation stability. Second, we introduce the extended Whitney homology of the lattice of set partitions, and show that it determines a finitely generated PD-module. It is known that the ordinary Whitney homology of the lattice of set partitions forms a finitely generated FI-module, and we are able to recover, and generalize this result.

This thesis is dedicated to my parents, Julie and Fawzi,

for always believing in me.

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CHAPTER 1

INTRODUCTION

The central object of this thesis is the representation theory of categories. That is, for a category $\mathcal{C}$, we study functors,

$$V : \mathcal{C} \to \mathsf{Vect},$$

where Vect is the category of vector spaces. Such functors form a category $\mathcal{C}$-Mod, which we informally refer to as the representation theory of $\mathcal{C}$. A single representation is called a $\mathcal{C}$-module.

As a first example, any group G can be viewed as a category G with a unique object and with (endo)morphisms labelled by $g \in G$. The representation theory of the category G coincides precisely with the usual representation theory of the group G by which we mean that an ordinary representation of G determines and is determined by a functor $V \in \mathsf{G}$-Mod (see Section 2.2.1). The 'atomic unit' of this thesis is the representation theory of symmetric groups S_n. It is natural to view the category associated to S_n as having a single object $\mathbf{n} := \{1, \ldots, n\}$ the finite set of n elements, and with (endo)morphisms given by the bijections $f : \mathbf{n} \to \mathbf{n}$. Functors from this category to Vect are precisely representations of S_n, and are well-studied (see Section 2.1.1). We view this category as atomic in the sense that we will construct larger categories from multiple copies of it. One well-studied example of this is the (skeleton of the) category FB, with one object $\mathbf{n}$ for each $n \in \mathbb{N}$, and with morphisms given by bijections. A functor $\mathsf{FB} \to \mathsf{Vect}$ is nothing more than a sequence of representations of symmetric groups S_n, one for each $n \in \mathbb{N}$. In their seminal paper [7], Church-Ellenberg-Farb observed that many interesting examples of representations of FB that appear throughout the literature enjoy a powerful representation theoretic constraint they call representation stability (see [7, 9]). This phenomenon can be seen as a representation theoretic analog of homological stability (e.g., [5, 15, 17]) and has many useful consequences, among them converting an *a priori* infinite amount of data (an infinite sequence of representations of S_n, one for each $n \in \mathbb{N}$), into a finite amount of data (see Section 2.3.1 for more details). Conceptually, you should care about representation stability because it appears in many disparate places throughout mathematics (it is ubiquitous) and because it imposes tight constraints on sequences of representations that might allow one to better understand the underlying object (it is desirable).

In [7] and [8] the authors completely characterize representation stability as a single structural property of a functor in the 'representations of categories' framework. Concretely, the category FB embeds into a larger category FI, with the same objects, but with morphisms given by injections (of which, bijections are a special case!). There is a notion of a finitely generated FI-module (see Definition 2.3.5), which the authors prove is equivalent to representation stability (see Theorem 2.3.9 for a precise statement). The advantage of this perspective is two-fold.

First, by providing a single concept one is able to simplify statements and proofs involving representation stability. There is a growing literature taking advantage of this approach to representation stability [6, 7, 8, 25, 27, 28, 29]. In Chapter 5 we add to this by providing another example of a finitely generated FI-module that appears in the study of free group automorphisms and their homology. Concretely, we study a sequence of cohomology spaces of certain groups $\Gamma_{n,s}$,

$$\{H^i(\Gamma_{n,s}) : s \in \mathbb{N}\},$$

The groups $\Gamma_{n,s}$ have appeared frequently in the study of free group automorphisms

[3, 10, 16, 17]. More recently, their cohomology $H^i(\Gamma_{n,s})$ was used in [10] to construct new, so-called unstable homology classes for $\mathrm{Out}(F_n)$. Their approach relied on the fact that the spaces $H^i(\Gamma_{n,s})$ admit an action of the symmetric group S_s. Building on this observation, we show in Chapter 3, for fixed $i, n \in \mathbb{N}$, the sequence $\{H^i(\Gamma_{n,s}) : s \in \mathbb{N}\}$ forms an FI-module. Our first main result is that the FI-module, which we denote $H^i(\Gamma_{n,\bullet})$, enjoys the property of finite generation (see Theorem 3.1.2).

Theorem A. *The* FI*-module* $H^i(\Gamma_{n,\bullet})$ *is finitely generated of stability degree* n *and weight* i.

Here, stability degree and weight are properties of an FI-module that enforce bounds on the underlying representation theory. Concretely, the sequence $H^i(\Gamma_{n,s})$ is said to satisfy representation stability (Definition 2.3.7), and the stability degree (Definition 2.3.10) and weight (Definition 2.3.11) control the stable range. In particular, we deduce the following as an immediate consequence (see Theorem 3.1.1).

Theorem B. *For fixed* i *and* n*, the sequence,*

$$\{H^i(\Gamma_{n,s}) : s \in \mathbb{N}\},$$

is uniformly representation stable with stable range $s \geq n + i$.

The second advantage of this perspective is that the framework of finitely generated representations of categories itself can be taken as a template and applied more generally. In Chapter 5 we introduce a new category PD and study its representation theory. Moreover, we show that the analogous notion of finite generation in the setting of PD-modules corresponds to a new representation theoretic constraint modeled on the representation stability of the one-dimensional setting. Concretely, we regard the category FB as a one-dimensional extension symmetric groups whose representation theory gives rise to sequences of S_n-modules with $n \in \mathbb{N}$. Similarly we consider the product category $\mathsf{FB} \times \mathsf{FB}$ as a two dimensional analog whose representation theory gives rise to arrays of $(\mathrm{S}_i, \mathrm{S}_n)$-bimodules with both $i \in \mathbb{N}$ and $n \in \mathbb{N}$. Just as the category FI was seen to be an enlargement of FB, so too the category PD can be seen as an enlargement of the product category $\mathsf{FB} \times \mathsf{FB}$. This point of view allows us to recast the notions of finite generation and representation stability (Definition 5.5.2) in this two-dimensional setting. In particular we prove (see Theorem 5.5.3 for a stronger and more precise statement),

Theorem C. *Let W be a finitely generated PD-module. Then W satisfies representation stability.*

We provide two detailed examples of finitely generated PD-modules. The first example arises as a family of coefficients $c_{\lambda\mu}$ indexed by pairs of partitions λ, μ arising from study of the Johnson homomorphism of the mapping class group, and in particular, from the n-th tensor power of the free associative algebra on a vector space $\mathcal{T}(V)^{\otimes n}$. In Chapter 4 we give an efficient algorithm computing these coefficients, and present visualizations that motivate our two-dimensional analog of representation stability. We are able to prove in Chapter 5 that these coefficients do indeed arise from a finitely generated PD-module. The following theorem is thus a corollary to Theorem C. See Theorem 5.6.5.

Theorem D. *The coefficients $c_{\lambda\mu}$ satisfy representation stability.*

Our second key example is discussed in Chapter 6 where we define a new combinatorial object, the extended Whitney homology, as a generalization of the Whitney homology of the lattice of set partitions. The extended Whitney homology $\widetilde{\mathcal{WH}}_{i,n}$ is a $(\mathrm{S}_i, \mathrm{S}_n)$-bimodule whose S_i-invariants recover the usual Whitney homology of the lattice of set partitions. We give a careful combinatorial description of this object, and show that it forms a finitely generated PD-module and thus its representation theory is stable. See Corollary 6.3.14 for a precise statement.

Theorem E. *The* $(\mathrm{S}_i, \mathrm{S}_n)$*-bimodules* $\widetilde{\mathcal{WH}}_{i,n}$ *satisfy representation stability.*

It is known that the ordinary Whitney homology of the lattice of set partitions forms a finitely generated FI-module [18]. We recover this result in Proposition 6.3.15.

In both of these examples we provide explicit computations. First we present an algorithm that computes the coefficients $c_{\lambda\mu}$. Our approach is to reinterpret these coefficients as counting the number of ways to solve certain *decomposition puzzles* (Fig. 1.1). This perspective allows the design of an efficient algorithm (Algorithm 4) extending the range of computation by a factor of over 750. In particular, we compute all coefficients $c_{\lambda\mu}$ for $|\lambda|, |\mu| < 15$, which amounts to 257,049 coefficients. Furthermore, we are able to visualize these computations by plotting a matrix (Fig. 1.2) whose (λ, μ) entry is colored according to the coefficient $c_{\lambda\mu}$. The patterns present in this data are striking and motivate the construction of the category PD. Full details of the algorithm and discussion of the data above are given in Chapter 4.

Our second computation is of the irreducible structure of the extended Whitney homology $\widetilde{\mathcal{WH}}$ of the lattice of set partitions, presented in Chapter 6. This new combinatorial object arises from the well-studied lattice of set partitions by extending the action to products of symmetric groups. We provide a detailed description of its underlying combinatorics in Section 6.2, and of its representation theory in Section 6.3. We present an explicit description of bimodule structure of $\widetilde{\mathcal{WH}}$ in terms of the twisted Lie operad $\widehat{\mathsf{Lie}}$ (see Theorem 6.2.19).

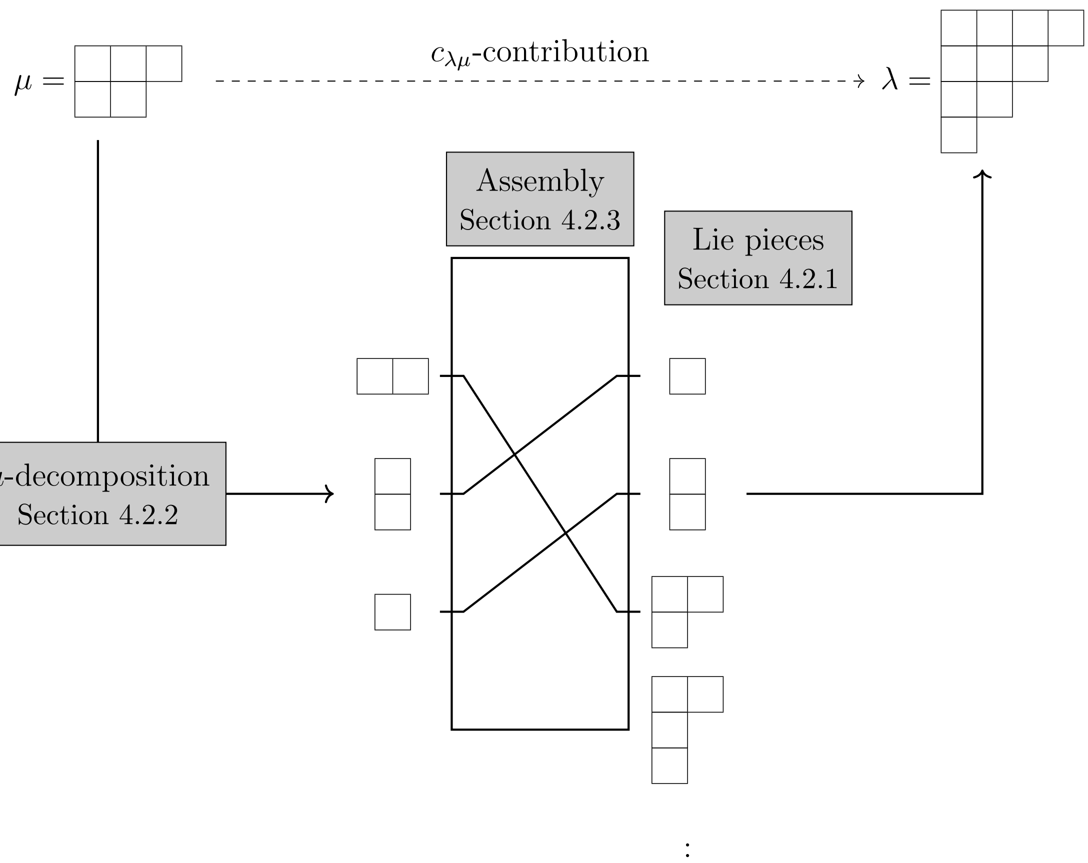


Figure 1.1: A schematic overview of a solution to a decomposition puzzle. See Section 4.2.2 for a complete description

Theorem F. *Fix $i, n \in \mathbb{N}$ and let $\lambda \vdash n$ be a partition of length i. There is an isomorphism of $(\mathrm{S}_i, \mathrm{S}_n)$-bimodules,*

$$\widetilde{\mathcal{WH}}_\lambda \cong \bigoplus_{\mu \vdash i} P_\mu \otimes P_\mu^\vee \left[\!\left[\widehat{\mathsf{Lie}} \right]\!\right]_\lambda .$$

The notation here is detailed in Chapter 6. As well as being consistent with the examples computed, and with the representation stability of Theorem E, this theorem can also be seen as a generalization of a result in [18]. In particular, in [18] Eq. (26) they state (albeit in the language of characters) that,

$$\mathcal{WH}_\lambda \cong \underbrace{\left(\circledast P_{(m_j)} \circ \widehat{\mathsf{Lie}}_j \right)}_{j \text{ odd}} \circledast \underbrace{\left(\circledast P_{(1^{m_j})} \circ \widehat{\mathsf{Lie}}_j \right)}_{j \text{ even}} .$$

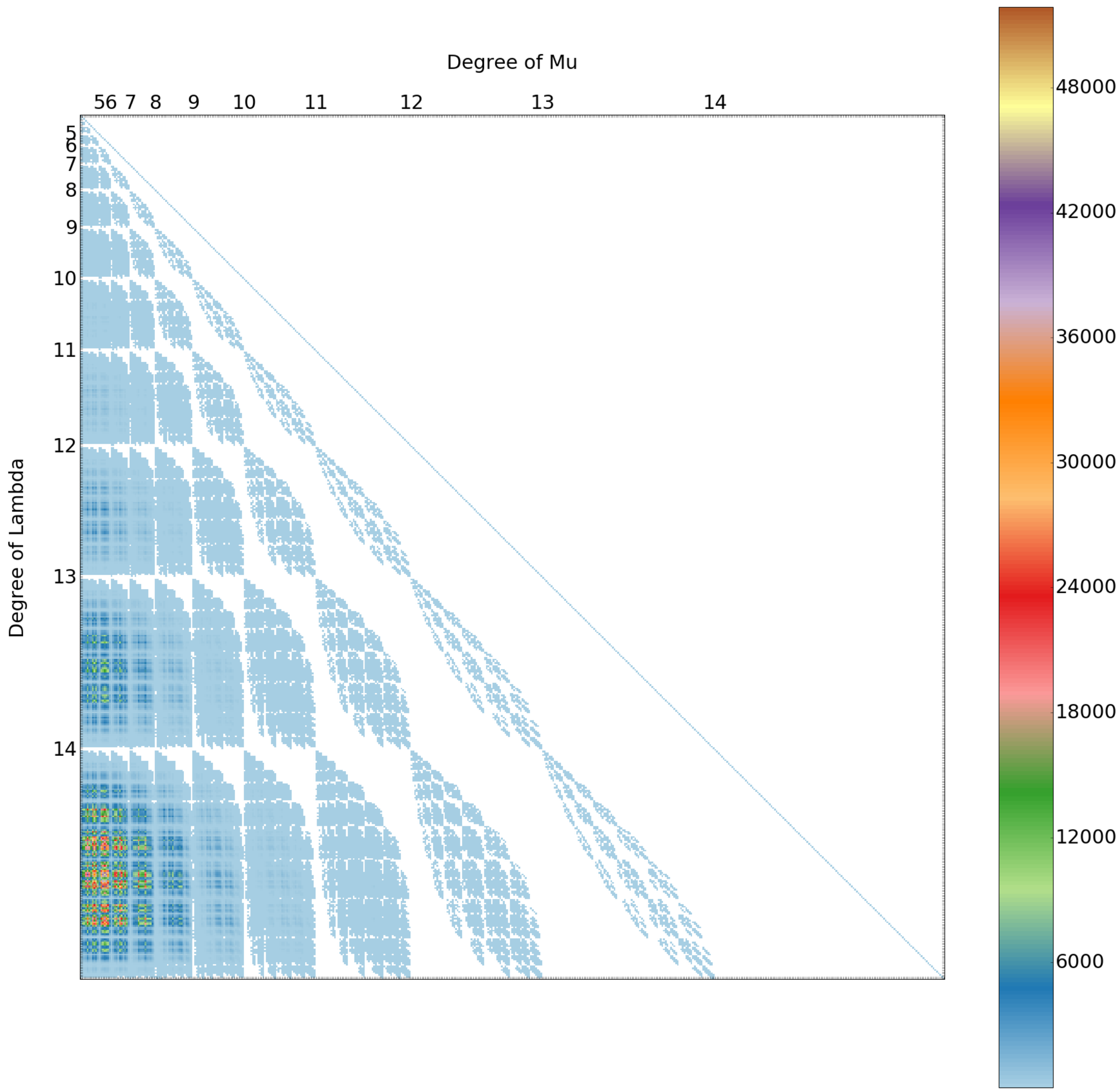


Figure 1.2: A visualization of all coefficients $c_{\lambda\mu}$ of degree < 15. These represents the full range of computations made by our algorithm. For readability we no longer label the partitions on the axes, instead we label the degree (or size) of the partitions at the point at which the degree changes.

This follows from taking the S_i-invariants of Theorem 6.2.19 (see Remark 6.2.21). Another application of this theorem allows us to bootstrap Algorithm 4 and compute the irreducible decomposition of $\widetilde{\mathcal{WH}}_{i,n}$ for $i, n < 11$ (see Remark 6.3.16) providing another visualization of a finitely generated PD-module.

CHAPTER 2

PRELIMINARIES

2.1 Representation theory of finite groups

We start by recording some basic notions from the representation theory of groups. Good references for this material include Brown [4], Etingof *et. al.* [12], Fulton-Harris [13] and James-Liebeck [19].

Fix a ground field $\Bbbk$ (of characteristic 0) throughout. Let G be a finite group. A vector space V (over $\Bbbk$) is a representation of G if there is a homomorphism $\rho : G \to \mathrm{GL}(V)$. Alternatively, consider the group ring $\Bbbk[G]$. That V is a representation of G is equivalent to the assertion that V is a left $\Bbbk[G]$-module. We thus interchangeably refer to such vector spaces V as representations and as $\Bbbk[G]$-modules. When the ground field $\Bbbk$ is understood, we simply refer to G-modules.

More often than not, the homomorphism ρ will be understood implicitly, and we will suppress the notation, writing $g \cdot v$ (or even gv when no ambiguity can arise) to denote $\rho(g)(v)$ for elements $g \in G$ and $v \in V$. In this case we will speak of defining the action of G on V.

A G-module homomorphism ϕ between representations V and W of G is a linear map $\phi : V \to W$ such that the following diagram commutes for all $g \in G$.

$$\begin{array}{ccc} V & \xrightarrow{\phi} & W \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ V & \xrightarrow{\phi} & W \end{array}$$

A subrepresentation of V is a vector subspace $W \subseteq V$ that is invariant under G. A representation V is called irreducible if its only non-zero subrepresentation is itself. A fundamental result in the representation theory of finite groups is complete-reducibility (also known as semisimplicity) which says that any representation V of G is a direct sum of irreducible representations.

Proposition 2.1.1 (e.g., Proposition 1.8, [13]). *For any representation V of a finite group G, there is a decomposition,*

$$V = V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k},$$

where the V_i are distinct irreducible representations of G. The decomposition of V into a direct sum of the k factors is unique, as are the V_i that occur as well as their multiplicities a_i.

This result tells us that to understand the representations of a finite group G, we should understand its irreducible representations. It is well-known that if G is a finite group then there are only finitely many non-isomorphic irreducible G-modules. In fact, the number of irreducible G-modules is exactly the number of conjugacy classes of G.

Combining representations. Let G and H be groups and consider the G-module V and the H-module W. We construct a representation $V \boxtimes W$ of $G \times H$ called the outer product of V and W as follows. As a vector space $V \boxtimes W$ is simply the tensor product $V \otimes W$. The action of $G \times H$ on $V \boxtimes W$ is as follows,

$$(g, h) \cdot v \boxtimes w := (g \cdot v) \boxtimes (h \cdot w),$$

where $g \in G, h \in H, v \in V$ and $w \in W$ and the actions $g \cdot v$ and $h \cdot w$ are determined by the G-module and H-module structure on V and W respectively. Here we are tacitly assuming that W is a left H-module, but we could equally well apply the above construction with a right action.

Bimodules. The interplay between left and right modules will play a significant role in what follows. Of particular interest will be when a vector space admits compatible left and right module structures. Concretely, if G and H are two groups, then a (G, H)-bimodule will mean a vector space V which is both a left G-module, a right H-module and satisfies $g \cdot (v \cdot h) = (g \cdot v) \cdot h$ for all $g \in G, h \in H$ and $v \in V$. Equivalently, a (G, H)-bimodule V is a left module over $G \times H$ in which the action of H on V is a right action.

Example 2.1.2. An important example is the group ring $\Bbbk[G]$ itself which is a (G, G)-bimodule with left and right actions given by left and right multiplication, respectively.

A useful result in this context describes the irreducible (G, H)-bimodules.

Lemma 2.1.3 (Theorem 4.25, [12])**.** *Let $V_1, \dots, V_n$ be a complete list of irreducible G-modules, and $W_1, \dots, W_m$ a complete list of irreducible H-modules. Then,*

$$\{V_i \boxtimes W_j : 1 \leq i \leq n, 1 \leq j \leq m\},$$

is a complete list of irreducible representations of $G \times H$. Equivalently, this is a complete list of irreducible (G, H)-bimodules if the W_j are understood to be right H-modules.

(Co)Invariants. If G is a group and V is a G-module, then the vector space of coinvariants V_G is the largest quotient of V on which G acts trivially. Concretely, it is the quotient of V under the subspace generated by elements of the form $v - g \cdot v$ for $g \in G$ and $v \in V$. The vector space of invariants V^G is the largest subspace of V on which G acts trivially, and so consists of elements $v \in V$ such that $g \cdot v = v$ for all $g \in G$. The following descriptions are well-known (see Brown [4]).

$$V_G \cong \Bbbk \otimes_G V, \qquad V^G \cong \mathrm{Hom}_G(\Bbbk, V),$$

where $\Bbbk$ is taken to be the G-module with trivial action of G.

2.1.1 Representation theory of symmetric groups

The central tool underlying all of the following theory is the representation theory of symmetric groups. We denote the symmetric group on n letters by S_n. A key property of representations of symmetric groups that facilitates all of these results is that they admit a uniform description in a common language. Concretely, the irreducible representations of S_n are in bijection with the set of partitions of n. This description allows us to meaningfully compare representations of different symmetric groups in a way we now make precise.

Definition 2.1.4. (Partitions.) A partition of n is a weakly decreasing sequence $\lambda = (\lambda_1, \ldots, \lambda_l)$ of positive integers λ_i such that $\sum_i \lambda_i = n$. We write $\lambda \vdash n$ to mean λ is a partition of n. We also write $|\lambda| = n$. For convenience, we also allow the empty partition $\varnothing$. To such a partition λ we associate its *Young diagram*, that is the collection of (left-justified) boxes, with λ_i boxes in the i-th row. For example, the partition $\lambda = (5, 3, 3, 1)$ has corresponding Young diagram,

$$\ydiagram{5,3,3,1}\ .$$

The *conjugate partition* of λ is denoted λ', and is the partition obtained by flipping the Young diagram along its main diagonal. Continuing with the example above we have,

$$\lambda = \ydiagram{5,3,3,1} \quad \rightsquigarrow \quad \lambda' = \ydiagram{4,3,3,1,1}\ ,$$

and $\lambda' = (4, 3, 3, 1, 1)$. A *tableau* T of shape λ is a numbering of the boxes in its Young diagram (bijectively) with the numbers $1, \ldots, n = |\lambda|$.

It is well-known (see, for example, Fulton-Harris [13]) that from a tableau T of shape $\lambda \vdash n$, a certain idempotent c_λ in the group algebra $\Bbbk[\mathrm{S}_n]$ called the *Young* *symmetrizer* can be used to construct an irreducible representation as follows,

$$P_\lambda := \Bbbk[\mathrm{S}_n] \cdot c_\lambda.$$

It turns out that this representation only depends on the underlying partition λ, and so we denote it by P_λ. Moreover, this construction gives every irreducible representation of symmetric groups. That is, the irreducible representations of S_n are in bijection with the partitions $\lambda \vdash n$. For example, for any $n \geq 0$, we have that $P_{(n)}$ is the 1-dimensional trivial representation of S_n and that $P_{(1^n)}$ is the 1-dimensional sign representation of S_n.

It will frequently be useful to denote the representation P_λ by the *Young diagram* for λ.

Yet another useful way to describe a partition will be in terms of its *exponents*. Write a partition λ as $1^{m_1}2^{m_2}\cdots$. We call m_i the exponents of λ. For example, $\lambda = (5,3,3,1)$ has exponents $m_1 = 1, m_3 = 2, m_5 = 1$ (and all other $m_i = 0$).

Twisted representations. Given two representations V, W of S_n, their tensor product $V \otimes W$ is also an S_n-module with diagonal action of S_n. As an example of this, take the sign representation $P_{(1^n)}$, which we often denote as ϵ_n. Given an S_n-module V, we form the S_n-module,

$$V \otimes \epsilon_n.$$

The representation V is said to have been twisted by the sign. It is well-known that for any $\lambda \vdash n$ we have,

$$P_\lambda \otimes \epsilon_n \cong P_{\lambda'},$$

where λ' is the conjugate partition of λ.

Padded partitions. In what follows we frequently compare representations of different symmetric groups using the following notation.

Definition 2.1.5. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$, define, for any $n \geq \lambda_1 + |\lambda|$, the *padded partition*

$$\lambda[n] = (n - |\lambda|, \lambda_1, \ldots, \lambda_l).$$

Accordingly, we denote by $P(\lambda)_n$ the irreducible S_n-module $P_{\lambda[n]}$ corresponding to the padded partition $\lambda[n]$.

In this way we find a notational similarity between the S_n-module $P(\lambda)_n$ and the S_m-module $P(\lambda)_m$. For example, the trivial representation of any symmetric group S_n corresponds to the padding of the empty partition $P(\varnothing)_n$. While this similarity in notation may seem like nothing more than a convenience at this point, we will see that it facilitates much of the theory that follows! This will first become evident when we discuss the theory of FI-modules (Section 2.3).

Bimodules over symmetric groups

We will frequently have cause to consider $(\mathrm{S}_i, \mathrm{S}_n)$-bimodules. By Lemma 2.1.3, we have that all irreducible $(\mathrm{S}_i, \mathrm{S}_n)$-bimodules are of the form,

$$P_\mu \boxtimes P_\lambda,$$

for partitions $\mu \vdash i, \lambda \vdash n$.

Example 2.1.6. (The group algebra, $\Bbbk[\mathrm{S}_n]$.)
The group algebra $\Bbbk[\mathrm{S}_n]$ is a $(\mathrm{S}_n, \mathrm{S}_n)$-bimodule. It is well-known to admit the following decomposition into irreducible bimodules.

$$\Bbbk[\mathrm{S}_n] \cong \bigoplus_{\lambda \vdash n} P_\lambda \boxtimes P_\lambda.$$

2.1.2 S-modules and Schur functors

Fix a vector space V throughout this section. Good references for material include Aguiar-Mahajan [1] and Loday-Vallette [22].

Definition 2.1.7. An $\mathbf{S}$-module (over $\Bbbk$) is a sequence,

$$M = (M(0), M(1), \ldots, M(n), \ldots),$$

of right $\Bbbk[\mathrm{S}_n]$-modules $M(n)$. We say that $M(n)$ is the degree n term of M. The twisted $\mathbf{S}$-module $\widehat{M}$ is the $\mathbf{S}$-module obtained from M by twisting each $M(n)$ by the sign representation. That is,

$$\widehat{M} = (M(0) \otimes \epsilon_0, M(1) \otimes \epsilon_1, \ldots, M(n) \otimes \epsilon_n, \ldots),$$

Fix a vector space V. Its n-th tensor power $V^{\otimes n}$ is a left S_n-module where S_n acts on the left by permuting the tensor factors. It can also be seen as a right $\mathrm{GL}(V)$-module, where $\mathrm{GL}(V)$ acts diagonally on the tensor factors (after inverting). Moreover, these actions are seen to commute, and so we have that $V^{\otimes n}$ is a $(\mathrm{S}_n, \mathrm{GL}(V))$-bimodule. In particular, if U is a right S_n-module, then,

$$U \otimes_{\mathrm{S}_n} V^{\otimes n},$$

is a right $GL(V)$-module, and, if W is a left $GL(V)$-module, then,

$$V^{\otimes n} \otimes_{\mathrm{GL}(V)} W,$$

is a left S_n-module.

Definition 2.1.8. Given an $\mathbf{S}$-module M, we associate its Schur functor,

$$\mathbb{S}_M(V) := \bigoplus_{n \geq 0} M(n) \otimes_{\mathrm{S}_n} V^{\otimes n}.$$

An S_n-module W can be considered as an $\mathbf{S}$-module with zeros everywhere except W in degree n. Its associated Schur functor is denoted,

$$\mathbb{S}_W(V) = W \otimes_{\mathrm{S}_n} V^{\otimes n}.$$

In this case the $\mathrm{GL}(V)$-module $\mathbb{S}_W(V)$ and the S_n-module W are said to be Schur-Weyl dual to one another.

Example 2.1.9.

1. (Tensor algebra.) The $\mathbf{S}$-module defined by $M(n) = \mathbb{k}[\mathrm{S}_n]$ for all $n \in \mathbb{N}$, determines the Schur functor,

$$\mathbb{S}_M(V) = \bigoplus_{n\geq 0} V^{\otimes n}.$$

This is the tensor algebra $\mathcal{T}(V)$, and is a graded vector space,

$$\mathcal{T}(V) = \bigoplus_{n\geq 0} \mathcal{T}_n(V),$$

where $\mathcal{T}_n(V) := V^{\otimes n}$. We denote the element $v_1 \otimes \cdots \otimes v_n$ of $\mathcal{T}_n(V)$ as $v_1 \cdots v_n$ when no ambiguity can arise. This is the free associative algebra on V and is easily seen to be a $\mathrm{GL}(V)$-module with diagonal action on the tensor factors. In particular, this action respects the grading, and so each graded piece $\mathcal{T}_n(V)$ is a $\mathrm{GL}(V)$-submodule of $\mathcal{T}(V)$.

2. (Symmetric algebra.) The $\mathbf{S}$-module defined by $M(n) = \mathbb{k}$ for all $n \in \mathbb{N}$ determines the Schur functor,

$$\mathbb{S}_M(V) = \bigoplus_{n\geq 0} \mathbb{k} \otimes_{\mathrm{S}_n} V^{\otimes n},$$

which is the symmetric algebra $\mathcal{S}(V)$. Indeed, for any $\sigma \in \mathrm{S}_n$, the tensors $v_1 \cdots v_n$ and $v_{\sigma(1)} \cdots v_{\sigma(n)}$ are identified under the tensor product $- \otimes_{\mathrm{S}_n} -$. This is again a graded vector space with $\mathcal{S}_n(V)$ corresponding to the n-th summand above.

In both of these examples we started from an $\mathbf{S}$-module and recovered a well-known algebra. We now present an example going in the other direction, starting with the free Lie algebra $\mathcal{L}(V)$ and recovering an $\mathbf{S}$-module Lie.

First, recall that a Lie algebra over $\Bbbk$ is a $\Bbbk$-vector space $\mathfrak{g}$ equipped with a bilinear map,

$$[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

called the *bracket*, satisfying, for all $x, y, z \in \mathfrak{g}$:

1. $[x, x] = 0$,
2. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$.

A homomorphism between Lie algebras $\mathfrak{g}, \mathfrak{g}'$ is a linear map $\phi : \mathfrak{g} \to \mathfrak{g}'$ such that,

$$\phi([x, y]) = [\phi(x), \phi(y)],$$

for all $x, y \in \mathfrak{g}$.

There are many formulations of the free Lie algebra $\mathcal{L}(V)$ on a vector space V. We record the following from [22].

Proposition 2.1.10 ([22], Proposition 13.2.3 (a))**.** *The following characterizes the free Lie algebra $\mathcal{L}(V)$ as a subspace of tensor algebra $\mathcal{T}(V)$. The subspace $\mathcal{L}(V) \subset \mathcal{T}(V)$ is generated by V under the bracket operation.*

Let V_n be the n-dimensional vector space with basis $\{x_1, \ldots, x_n\}$. Let Lie_n denote the subspace of $\mathcal{L}(V_n)$ which is linear in each variable x_i. For example, Lie_1 is one dimensional and is spanned by x_1, and Lie_2 is one dimensional and is spanned by $[x_1, x_2]$. In general, the n-th subspace Lie_n admits an action of S_n, giving rise to an $\mathbf{S}$-module.

Definition 2.1.11. Let Lie be the $\mathbf{S}$-module,

$$\mathsf{Lie} = (\mathsf{Lie}_0, \mathsf{Lie}_1, \ldots, \mathsf{Lie}_n, \ldots)$$

The associated Schur functor $\mathbb{S}_{\mathsf{Lie}}(V)$ is precisely the free Lie algebra $\mathcal{L}(V)$,

$$\mathcal{L}(V) \cong \bigoplus_{n \geq 0} \mathsf{Lie}_n \otimes_{\mathrm{S}_n} V^{\otimes n},$$

and the n-th graded piece $\mathcal{L}_n(V)$ is Schur-Weyl dual to the S_n-module Lie_n [22].

Let C_n denote the cyclic subgroup of S_n generated by a n-cycle c, and let ζ denote a(ny) one-dimensional (complex) representation sending c to a primitive n-th root of unity. As a (complex) representation of S_n it can be shown (e.g. [20]) that Lie_n isomorphic to the induced module,

$$\mathrm{Ind}_{C_n}^{\mathrm{S}_n} \zeta.$$

Using this description one can compute the decomposition into irreducible S_n-modules of Lie_n. For example,

$$\mathsf{Lie}_0 \cong \varnothing, \qquad \mathsf{Lie}_1 \cong \ydiagram{1}\,, \qquad \mathsf{Lie}_2 \cong \ydiagram{1,1} \qquad \mathsf{Lie}_3 = \ydiagram{2,1}\,,$$

$$\mathsf{Lie}_4 \cong \ydiagram{3,1} \oplus \ydiagram{2,1,1}\,,$$

$$\mathsf{Lie}_5 \cong \ydiagram{4,1} \oplus \ydiagram{3,2} \oplus \ydiagram{3,1,1} \oplus \ydiagram{2,2,1} \oplus \ydiagram{2,1,1,1}\,.$$

The twisted counterpart to Lie will play a significant role in our story. For example we have

$$\mathsf{Lie}_2 \cong \ydiagram{1,1} \quad \rightsquigarrow \quad \widehat{\mathsf{Lie}}_2 \cong \ydiagram{2}$$

Schur-Weyl duality theorem.

Definition 2.1.12. The Schur functor associated to the irreducible S_n-module P_λ is denoted simply,

$$\mathbb{S}_\lambda(V) := P_\lambda \otimes_{\mathrm{S}_n} V^{\otimes n}.$$

The Schur-Weyl duality theorem gives a decomposition of $V^{\otimes n}$ into irreducible bimodules ([12], Corollary 4.59).

Theorem 2.1.13 (The Schur-Wely Duality Theorem.)**.** *As a* $(\mathrm{S}_n, \mathrm{GL}(V))$*-bimodule,* $V^{\otimes n}$ *decomposes as,*

$$V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} P_\lambda \boxtimes \mathbb{S}_\lambda(V),$$

where $\mathbb{S}_\lambda(V)$ *are distinct irreducible representations of* $\mathrm{GL}(V)$ *or zero.*

2.1.3 Induction of representations

Let H be a subgroup of G. Then any G-module V can be regarded as an H-module via the inclusion $\alpha : H \hookrightarrow G$. That is, for $h \in H$ and $v \in V$, define,

$$h \cdot v := \alpha(h) \cdot v.$$

For any G-module V, we denote by $\mathrm{Res}_H^G V$ the H-module obtained in this way. This construction is referred to as restriction.

Going in the other direction is the operation of induction, which turns H-modules into G-modules. Concretely, for any H-module W, we define the induction from H to G of W as,

$$\mathrm{Ind}_H^G W := \Bbbk[G] \otimes_H W,$$

where $\Bbbk[G]$ is regarded as a right $\Bbbk[H]$-module as follows: for $x \in \Bbbk[G]$ and $h \in H$ define the right action of H via $x \cdot h := x\alpha(h)$. By associativity, the left action of $\Bbbk[G]$

on itself commutes with this right action, and thus $\mathrm{Ind}_H^G W$ is can be made a G-module via,

$$g \cdot (x \otimes w) := gx \otimes w,$$

where $w \in W, x \in \Bbbk[G]$ and $g \in G$.

We now present a useful characterization of induced modules. Although this presentation is well-known (see [4], for example), we present its proof as its methods foreshadow many of the techniques we will use throughout.

Proposition 2.1.14 (Proposition 5.1, [4]). *The G-module $\mathrm{Ind}_H^G W$ contains W as an H-submodule, and is the direct sum of its transforms,*

$$\mathrm{Ind}_H^G W = \bigoplus_{g \in G/H} gW.$$

Proof. Let E be a set of (left) coset representatives for G/H. Then $\Bbbk[G]$ is a free (right) H-module where we can take E as a basis. We therefore have the following decomposition on the level of vector spaces,

$$\mathrm{Ind}_H^G W = \bigoplus_{g \in E} g \otimes W,$$

where $g \otimes W := \{g \otimes w : w \in W\}$ is an isomorphic copy of W. Furthermore, by taking 1 as the representative for its coset, we have a canonical inclusion,

$$i : W \hookrightarrow \mathrm{Ind}_H^G W, \tag{2.1}$$

defined by $w \mapsto 1 \otimes w$. This is a map of H-modules, where $\mathrm{Ind}_H^G W$ is regarded as an H-module via restriction. Indeed, let $w \in W$ and $h \in H$. By definition,

$$1 \otimes h \cdot w = \alpha(h) \otimes w = \alpha(h) \cdot (1 \otimes w),$$

in $W \otimes_{\Bbbk[H]} \Bbbk[G]$, and so $i(h \cdot w) = \alpha(h) \cdot i(w)$, as required. Finally, the summand $g \otimes W$ can be seen as the transform of W under the action of $g \in G$ via,

$$g \cdot (1 \otimes w) = g \otimes w,$$

in $W \otimes_{\Bbbk[H]} \Bbbk[G]$. □

This result completely characterizes induced G-modules. In particular, we have the following.

Lemma 2.1.15 (Proposition 5.3, [4]). *Let V be a G-module whose underlying vector space is a direct sum,*

$$V = \bigoplus_{i \in I} W_i.$$

Further, assume that the G-action transitively permutes the summands of V. Fix a summand $W = W_i$ and let H be the isotropy group of i. Then W is an H-module and,

$$V \cong \operatorname{Ind}_H^G W.$$

It is useful to note here that it is possible to compose inductions and restrictions.

Proposition 2.1.16. *Let $K \leq H$ be subgroups of G and let U be a K-module. There is an isomorphism of G-modules,*

$$\operatorname{Ind}_K^G U \cong \operatorname{Ind}_H^G \operatorname{Ind}_K^H U.$$

Similarly, for a G-module V, there is an isomorphism of K-modules,

$$\operatorname{Res}_K^G V \cong \operatorname{Res}_K^H \operatorname{Res}_H^G V.$$

Proof. Let E be a set of coset representatives for G/H, and F be a set of coset representatives for H/K. Then $\{gh : g \in E, h \in F\}$ is a set of coset representatives for G/K. Write,

$$W = \operatorname{Ind}_K^H U = \bigoplus_{h \in F} hU.$$

Then,

$$\mathrm{Ind}_H^G(W) = \bigoplus_{g \in E} W = \bigoplus_{g \in E, h \in F} ghU = \mathrm{Ind}_K^G U.$$

The parallel statement for restriction is immediate. $\square$

Example: Symmetric groups

We start by defining the induction product for symmetric groups. Given d non-negative integers $k_1, \dots, k_d$ and representations V_i of S_{k_i} we construct a representation,

$$V_1 \circledast \cdots \circledast V_d,$$

of S_n, where $n := \sum_i k_i$ as the induction to S_n of the outer product,

$$V_1 \boxtimes \cdots \boxtimes V_d,$$

a representation of $\mathrm{S}_{k_1} \times \cdots \times \mathrm{S}_{k_d}$. That is,

$$V_1 \circledast \cdots \circledast V_d := \mathrm{Ind}_{\mathrm{S}_{k_1} \times \cdots \times \mathrm{S}_{k_d}}^{\mathrm{S}_n} V_1 \boxtimes \cdots \boxtimes V_d.$$

This product is both commutative and associative (see Fulton-Harris [13], Section 4).

It will frequently be useful to have a rule for decomposing such a representation into irreducibles. To that end, define the Littlewood-Richardson coefficients $a_{\lambda\mu}^{\nu}$ as the number of times the irreducible P_ν appears in $P_\lambda \circledast P_\mu$. There is combinatorial procedure, called the Littlewood-Richardson rule that determines the coefficient $a_{\lambda\mu}^{\nu}$ (see Fulton-Harris [13], Appendix A.8). We state a useful special case of this rule known as branching which addresses the case $\mu = (k)$ corresponding to the trivial S_k-module $P_{(k)}$.

Lemma 2.1.17 (Branching rules.)**.** *Let $\lambda \vdash n$ a partition, and P_λ the corresponding irreducible S_n-module.*

1. *The induction of the* $\mathrm{S}_n \times \mathrm{S}_k$*-module* $P_\lambda \boxtimes P_{(k)}$ *to an* S_{n+k} *module decomposes as,*

$$P_\lambda \circledast P_{(k)} = \mathrm{Ind}_{\mathrm{S}_n \times \mathrm{S}_k}^{\mathrm{S}_{n+k}} P_\lambda \boxtimes P_{(k)} \cong \bigoplus_{\mu} P_\mu,$$

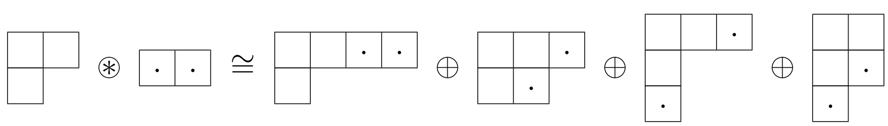
where the sum is over all partitions $\mu \vdash n+k$ *obtained from* λ *by adding one box to* k *different columns.*

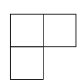
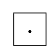
2. *The* S_k*-coinvariants of the restriction of the the* S_n*-module* P_λ *to a* $\mathrm{S}_{n-k} \times \mathrm{S}_k$*-module decomposes as,*

$$\left(\mathrm{Res}_{\mathrm{S}_{n-k} \times \mathrm{S}_k}^{\mathrm{S}_n} P_\lambda\right)_{\mathrm{S}_k} \cong \bigoplus_{\mu} P_\mu,$$

where the sum is over all partitions $\mu \vdash n-k$ *obtained from* λ *by removing one box from* k *different columns.*

Example 2.1.18. Denoting irreducible representations of S_n by their corresponding Young diagrams, we have the following isomorphism of S_4-modules as an example of the branching rule.



Here the sum is over partitions μ obtained from  by adding one box  to two different columns.

2.1.4 Inductions involving products of bimodules.

We discuss a special case of induction that will crop up frequently. Let $S(a,b) := \mathrm{S}_a \times \mathrm{S}_b$ denote the product of two symmetric groups. By Lemma 2.1.3, the irreducible representations of $S(a,b)$ are of the form,

$$P_\mu \boxtimes P_\lambda$$

for $\mu \vdash a$ and $\lambda \vdash b$. Consider now the product $S(a,b) \times S(c,d)$. Another application of Lemma 2.1.3 tells us that its irreducible representations are of the form,

$$(P_\lambda \boxtimes P_\mu) \boxtimes (P_\nu \boxtimes P_\eta),$$

for $\mu \vdash a, \lambda \vdash b, \nu \vdash c$ and $\eta \vdash d$.

We are interested induction defined by the subgroup,

$$S(a,b) \times S(c,d) \leq S(a+c, b+d),$$

which can be described in terms of its irreducible $S(a,b) \times S(c,d)$ modules as,

$$\mathrm{Ind}_{S(a,b)\times S(c,d)}^{S(a+b,c+d)} (P_\mu \boxtimes P_\lambda) \boxtimes (P_\nu \boxtimes P_\eta) \cong \left(\mathrm{Ind}_{S(a,c)}^{\mathrm{S}_{a+c}} P_\mu \boxtimes P_\nu\right) \boxtimes \left(\mathrm{Ind}_{S(b,d)}^{\mathrm{S}_{b+d}} P_\lambda \boxtimes P_\eta\right), \tag{2.2}$$

using the isomorphism,

$$S(a,b) \times S(c,d) \cong S(a,c) \times S(b,d).$$

Example 2.1.19. Consider the irreducible $\mathrm{S}_2 \times \mathrm{S}_1$-module,

$$\ydiagram{1,1} \boxtimes \ydiagram{1}$$

and the irreducible $\mathrm{S}_2 \times \mathrm{S}_2$-module,

$$\ydiagram{2} \boxtimes \ydiagram{1,1}\,.$$

We have,

$$\mathrm{Ind}_{S(2,1)\times S(2,2)}^{S(4,3)} \left(\ydiagram{1,1} \boxtimes \ydiagram{1}\right) \boxtimes \left(\ydiagram{2} \boxtimes \ydiagram{1,1}\right) \cong$$

$$\left(\mathrm{Ind}_{S(2,2)}^{\mathrm{S}_4} \ydiagram{1,1} \boxtimes \ydiagram{2}\right) \boxtimes \left(\mathrm{Ind}_{S(1,2)}^{\mathrm{S}_3} \ydiagram{1} \boxtimes \ydiagram{1,1}\right) \cong$$

$$\left(\ydiagram{3,1} \oplus \ydiagram{2,1,1}\right) \boxtimes \left(\ydiagram{2,1} \oplus \ydiagram{1,1,1}\right).$$

2.1.5 Plethysm

Denote the (permutational) wreath product of S_n with S_m by $\mathrm{S}_n[\mathrm{S}_m]$. Recall that this is the normalizer subgroup of $(\mathrm{S}_m)^n$ in S_{mn}, or equivalently it is the semidirect product $\mathrm{S}_n \ltimes (\mathrm{S}_m)^n$ where S_n acts on $(\mathrm{S}_m)^n$ by permuting coordinates. Given an S_n-module V and an S_m-module W, let,

$$V[W] := V \otimes (W^{\otimes n}). \tag{2.3}$$

Then $V[W]$ admits the structure of an $\mathrm{S}_n[\mathrm{S}_m]$-module, where,

- $(\mathrm{S}_m)^n$ acts componentwise on $W^{\otimes n}$, and,
- S_n acts simultaneously on V and by permuting the tensor factors in $W^{\otimes n}$.

The plethysm is the induced S_{mn}-module,

$$V \circ W := \mathrm{Ind}_{\mathrm{S}_n[\mathrm{S}_m]}^{\mathrm{S}_{mn}} V \otimes (W^{\otimes n}).$$

The plethysm problem is to decompose,

$$P_\lambda[P_\mu] = \bigoplus_\nu m_{\lambda\mu}^\nu P_\nu, \tag{2.4}$$

for partitions $\lambda \vdash n, \mu \vdash m$ and $\nu \vdash nm$. Computing the coefficients $m_{\lambda\mu}^\nu$ is an important general problem in representation theory [23].

Remark 2.1.20. The computer algebra package `SAGE` can compute $m_{\lambda\mu}^\nu$ for small partitions. For example,

$$\ydiagram{1,1} \circ \ydiagram{2} = \ydiagram{3,1},$$

$$\ydiagram{2} \circ \ydiagram{3} \cong \ydiagram{4,2} \oplus \ydiagram{6}.$$

2.1.6 Character theory

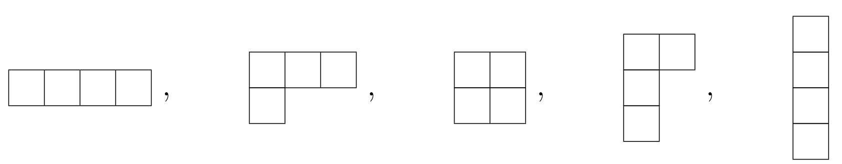
Associated to a representation $\rho : G \to \mathrm{GL}(V)$ is its character,

$$\begin{aligned} \chi : G &\to \Bbbk \\ g &\mapsto \mathrm{Tr}(\rho(g)) \end{aligned}$$

that assigns to each element of G the trace of its corresponding linear map under ρ. We record some basic properties of characters of representations. A good reference for this material is [19]. For convenience, we temporarily fix our ground field $\Bbbk = \mathbb{C}$.

Suppose that V and W are G-modules with characters χ_V and χ_W, respectively. Then V and W are isomorphic as G-modules if and only if their characters are equal $\chi_V = \chi_W$. Given a complete set of irreducible G-modules, $V_1, \ldots, V_n$, denote their characters, $\chi_1, \ldots, \chi_n$. These are the irreducible characters of G. The character table of G is a means to record each irreducible character. It is easy to see that characters are class functions (see [19] for a definition), and it will thus suffice to give the characters of each conjugacy class of G. We present a simple example.

Example 2.1.21. (Character table of S_4.)
There are five conjugacy classes of elements in S_4: $1, (12), (12)(34), (123)$ and (1234), and thus five irreducible S_4-modules,



We denote their characters $\chi_{(4)}, \chi_{(3,1)}, \chi_{(2,2)}, \chi_{(2,1,1)}$ and $\chi_{(1,1,1,1)}$, respectively. The character table for S_4 is:

	1	(12)	(12)(34)	(123)	(1234)
$\chi_{(4)}$	1	1	1	1	1
$\chi_{(1,1,1,1)}$	1	-1	1	1	-1
$\chi_{(3,1)}$	3	1	-1	0	-1
$\chi_{(2,1,1)}$	3	-1	-1	0	1
$\chi_{(2,2)}$	2	0	2	-1	0

Notice that the character of 1 records the dimension of the representation.

The character table provides a useful means for decomposing an arbitrary representation into its irreducible pieces. We now briefly describe this process. The space of complex-valued class functions on any finite group G has a natural inner product,

$$\langle f, f' \rangle_G := \frac{1}{|G|} \sum_{g \in G} f(g)\overline{f'(g)}.$$

Since f is a class function we can rewrite this sum as,

$$\langle f, f' \rangle_G := \frac{1}{|G|} \sum_{g \in \mathcal{C}} |\mathcal{C}(g)| f(g)\overline{f'(g)},$$

where $\mathcal{C}$ is the set of conjugacy classes in G, and $|\mathcal{C}(g)|$ is the size of the corresponding conjugacy class. If the group G is understood we denote this inner product simply as $\langle -, - \rangle$.

The inner products of characters of representations is related to the space of G-module homomorphisms between representations. Concretely,

Proposition 2.1.22 (Theorem 14.24, [19]). *Let V and W be G-modules with characters χ_V and χ_W, respectively. Then,*

$$\dim \operatorname{Hom}_G(V, W) = \langle \chi_V, \chi_W \rangle.$$

The fact that $\dim \mathrm{Hom}_G(\bigoplus_i V_i, \bigoplus_j W_j) = \sum_{i,j} \dim \mathrm{Hom}_G(V_i, W_j)$ combined with Schur's lemma, which says that $\dim \mathrm{Hom}_G(V_i, V_j) = \delta_{ij}$ if V_i, V_j are irreducible G-modules, gives us the following tool for computing decompositions into irreducible G-modules.

Lemma 2.1.23. *(Orthogonality relations.) Let $V_1, \dots, V_n$ be a complete list of irreducible G-modules, with corresponding characters $\chi_1, \dots, \chi_n$. Let W be a G-module with corresponding character ψ.*

1. *Then $\langle \psi, \chi_i \rangle$ is equal to the multiplicity with which V_i appears in W.*
2. *We have the decomposition,*

$$W = \bigoplus_{i=1}^{n} V_i^{\oplus \langle \psi, \chi_i \rangle},$$

of W into irreducible G-modules.

The orthogonality relations provide a technique for computing the decomposition into irreducible G-modules that we will use in several places. It is therefore worth providing a simple example.

Example 2.1.24. Consider the permutation representation $\rho : \mathrm{S}_3 \to \mathrm{GL}(\Bbbk^3)$ where S_3 acts by permuting the basis elements. We have the following descriptions,

$$\rho(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \rho((12)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \rho((123)) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

with corresponding characters $\psi(1) = 3, \psi(12) = 1$ and $\psi(123) = 0$. The character table for S_3 is,

	1	(12)	(123)
$\chi_{(3)}$	1	1	1
$\chi_{(1,1,1)}$	1	-1	1
$\chi_{(2,1)}$	2	0	-1

We use the orthogonality relations to compute,

$$\langle \psi, \chi_{(3)} \rangle = \frac{1}{6}\left(1 \cdot 3 \cdot 1 + 3 \cdot 1 \cdot 1 + 0\right) = 1$$

$$\langle \psi, \chi_{(1,1,1)} \rangle = \frac{1}{6}\left(1 \cdot 3 \cdot 1 + 3 \cdot 1 \cdot (-1) + 0\right) = 0$$

$$\langle \psi, \chi_{(2,1)} \rangle = \frac{1}{6}\left(1 \cdot 3 \cdot 2 + 0 + 0\right) = 1$$

and hence the decomposition,

$$\mathbb{k}^3 \cong \ydiagram{3} \oplus \ydiagram{2,1}\,.$$

Notice that the dimensions agree:

$$\dim\left(\ydiagram{3} \oplus \ydiagram{2,1}\right) = \dim\left(\ydiagram{3}\right) + \dim\left(\ydiagram{2,1}\right) = 1+2 = 3 = \dim\left(\mathbb{k}^3\right).$$

Frobenius reciprocity. The language of characters also enables us to state a powerful duality between induced and restricted representations. Let $H \leq G$ be groups, let V be a G-module, and let W an H-module. Denote the character of V by χ and the character of W by ψ. Induction and restriction can be applied to characters. In particular, let $\operatorname{Ind}_H^G \psi$ be the character corresponding to the representation $\operatorname{Ind}_H^G W$. Similarly, let $\operatorname{Res}_H^G \chi$ denote the character of the representation $\operatorname{Res}_H^G V$. Frobenius reciprocity can be stated as a relationship between these induced and restricted characters. This result is well-known (see, [19], for example).

Theorem 2.1.25 (The Frobenius Reciprocity Theorem)**.** *Let $H \leq G$ and let χ and ψ be as above. Then,*

$$\langle \operatorname{Ind}_H^G \psi, \chi \rangle_G = \langle \psi, \operatorname{Res}_H^G \chi \rangle_H.$$

2.2 Category theory

A lot of what follows is phrased in the language of categories. The following quote is taken from from Weibel [33]:

> The word "category" is due to Eilenberg and MacLane (1947) but was taken from Aristotle and Kant. It is chiefly used as an organizing principle for familiar notions.

That sentiment will hold firm here, where we will recast much of the theory above in the language of categories. This reorganization will be the foundation upon which our theory is built. We recall the basic notions here. See [24] for futher details.

Given a category C, we let $\mathrm{ob}(\mathsf{C})$ denote its objects, and for two such objects A, B we denote by $\mathrm{Hom}_{\mathsf{C}}(A, B)$ the morphisms $A \to B$ in C. Given objects $A, B, C \in \mathrm{ob}(\mathsf{C})$ and morphisms $\phi \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ and $\psi \in \mathrm{Hom}_{\mathsf{C}}(B, C)$, composition defines a morphism in the category,

$$\psi \circ \phi \in \mathrm{Hom}_{\mathsf{C}}(A, C).$$

A subcategory $\mathsf{D} \subseteq \mathsf{C}$ is a category whose objects are a subcollection of $\mathrm{ob}(\mathsf{C})$, and whose morphisms are a subcollection of morphisms of C, and where composition and identities are inherited from C. We say a subcategory is full if,

$$\mathrm{Hom}_{\mathsf{D}}(A, B) = \mathrm{Hom}_{\mathsf{C}}(A, B),$$

for all $A, B \in \mathrm{ob}(\mathsf{D})$. A category is skeletal if no two distinct objects are isomorphic. Given a category C, a skeleton of C is an equivalent category that is skeletal (see [24] for more details). The opposite of a category C is the category with the same objects but with all the arrows reversed. We denote this category C^{op}.

An important and ubiquitous concept in category theory is that of an adjunction between categories.

Definition 2.2.1. Two functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ between categories C, D are called adjoint functors if there is a natural bijection,

$$\mathrm{Hom}_{\mathsf{D}}(F(A), B) \cong \mathrm{Hom}_{\mathsf{C}}(A, G(B)),$$

for all $A \in \mathrm{ob}(\mathsf{C}), B \in \mathrm{ob}(\mathsf{D})$. We say F is left-adjoint to G, that G is right-adjoint to F, and we write,

$$F : \mathsf{C} \leftrightharpoons \mathsf{D} : G.$$

We introduce some important examples of categories that we will return to throughout the text.

Example 2.2.2.

1. (Basic examples.) Let Set denote the category of sets, whose objects are sets and whose morphisms are functions. Let $\mathsf{Vect}_{\Bbbk}$ denote the category of $\Bbbk$-vector spaces, whose objects are $\Bbbk$-vector spaces and whose morphisms are linear maps. When the ground field $\Bbbk$ is understood we denote this simply as Vect.

   A useful functor is the linearization functor $L : \mathsf{Set} \to \mathsf{Vect}$ that sends a set S to the free $\Bbbk$-vector space spanned by S. Given an arbitrary functor $F : \mathsf{C} \to \mathsf{Set}$, we define its linearization as the composition with $L \circ F : \mathsf{C} \to \mathsf{Vect}$.

2. (Groups as categories.) Given a group G we consider the category G with one object $\bullet$, and with morphisms $\mathrm{Hom}_{\mathsf{G}}(\bullet, \bullet) = G$. If $H \leq G$ is a subgroup then its corresponding category H is a subcategory of G.

3. (Finite sets and bijections.) Let FB denote the category whose objects are finite sets, and whose morphisms are bijections. That is, given sets $S, T \in \mathrm{ob}(\mathsf{FB})$, we have that,

$$\mathrm{Hom}_{\mathsf{FB}}(S,T) = \begin{cases} \mathrm{Sym}(S,T) & |S| = |T| \\ \varnothing & \text{else} \end{cases}$$

By assigning to the natural number $n \in \mathbb{N}$, the finite set $\mathbf{n} := \{1, \dots, n\}$, we realize the skeleton of FB as having objects natural numbers, and where the only morphisms are of the form $\mathrm{Hom}_{\mathsf{FB}}(\mathbf{n}, \mathbf{n}) = \mathrm{S}_n$.

4. (Finite sets and injections.) Let FI denote the category whose objects are finite sets and whose morphisms are injections. Notice that FB is a subcategory of FI. We can similarly define the skeleton of FI as having objects of the form $\mathbf{n}$ as above.

5. (Products of categories.) Let C and D be two categories. Then the product category $\mathsf{C} \times \mathsf{D}$ is the category whose objects are pairs (A, B) where $A \in \mathrm{ob}(\mathsf{C})$ and $B \in \mathrm{ob}(\mathsf{D})$, and whose morphisms are pairs (f, g) where f is a morphism in C and g a morphism in D. A functor from a product category is called a bifunctor.

   For example, the product $\mathsf{FB} \times \mathsf{FB}$ has objects pairs of finite sets (S, T). A morphism,

$$(S,T) \to (S',T'),$$

   in $\mathsf{FB} \times \mathsf{FB}$ is of the form (f, g) where $f \in \mathrm{Hom}_{\mathsf{FB}}(S, S')$ and $g \in \mathrm{Hom}_{\mathsf{FB}}(T, T')$. In particular, we require $|S| = |S'|$ and $|T| = |T'|$ in order for (f, g) to be non-zero.

2.2.1 Representations of categories

The prototypical example is that of representations of groups, or in other words, G-modules. Thinking of groups as categories as in Example 2.2.2(2) we see that a G-module V is exactly a functor,

$$\mathcal{V} : \mathsf{G} \to \mathsf{Vect}.$$

Indeed, the single object $\bullet \in \mathrm{ob}(\mathsf{G})$ is sent to the vector space V. Moreover, each element $g \in G$ defines a linear map $\rho(g) \in \mathrm{GL}(V)$, and this is precisely the image of the functor $\mathcal{V}$ on the endomorphisms labelled by g. We find the following picture instructive.

$$\begin{array}{ccc}
g \in G & & \rho(g) \in \mathrm{GL}(V) \\
\circlearrowright & & \circlearrowright \\
\bullet & & V \\
 & & \\
\mathsf{G} & \xrightarrow[\mathcal{V}]{\qquad\qquad} & \mathsf{Vect}
\end{array}$$

Remark 2.2.3. We will often merge these notations, and refer to the functor $\mathcal{V}$ as V.

It is routine to verify that the functors $\mathsf{G} \to \mathsf{Vect}$ form a category, whose morphisms are natural transformations. We denote this category $\mathsf{G}\text{-}\mathsf{Mod}$. Our philosophy is that the study of the category of G-modules is equivalent to the representation theory of G. In particular, we have the following dictionary.

Representation theory	Category of modules
Group G	Category G
Left G-module V	$V \in \mathsf{G}\text{-}\mathsf{Mod}$
Right G-module W	$W \in \mathsf{G}^{\mathsf{op}}\text{-}\mathsf{Mod}$
G-module homomorphism $\phi : V_1 \to V_2$	$\phi \in \mathrm{Hom}_{\mathsf{G}\text{-}\mathsf{Mod}}(V, V')$

A natural generalization, then, is to consider C-Mod, the category of functors from C to Vect for an arbitrary category C. We take the perspective that the study of C-Mod should be thought of as studying the representation theory of C by analogy with groups.

Example 2.2.4. The category FB-Mod consists of functors $W : \mathsf{FB} \to \mathsf{Vect}$. Considering the skeleton of FB, the functor W assigns to each $n \in \mathbb{N}$ a vector space W_n. Moreover, for each $f \in \mathrm{Hom}_{\mathsf{FB}}(n, n)$ there is a linear map $f_* : W_n \to W_n$. Functoriality of W amounts to insisting that W_n inherits the structure of an S_n-module, for each $n \in \mathbb{N}$. Thus an FB-module W is exactly an $\mathbf{S}$-module $W = (W_0, W_1, \ldots, W_n, \ldots)$. We find the following picture instructive.

$$
\begin{array}{ccc}
\vdots & & \vdots \\
\mathrm{S}_m \circlearrowright \mathbf{m} & & W_m \circlearrowleft \mathrm{S}_m \\
\vdots & & \vdots \\
\mathrm{S}_n \circlearrowright \mathbf{n} & & W_n \circlearrowleft \mathrm{S}_n \\
\vdots & & \vdots \\
\mathsf{FB} & \xrightarrow[W]{\qquad\qquad\qquad} & \mathsf{Vect}
\end{array}
$$

The representation theory of FI provides another important example. The category FI-Mod has been extensively since the seminal papers of Church-Ellenberg-Farb [7] and

Church-Ellenberg-Farb-Nagpal [8]. We recall that theory Section 2.3.

A fundamental notion in the theory of group representations (and in the study of R-modules more generally) is that of a bimodule. We make the appropriate generalization to this setting.

Definition 2.2.5. (Bimodules over product categories.) Let C and D be categories. Then a (C, D)-bimodule W is a bifunctor over the product category $\mathsf{C} \times \mathsf{D}$, i.e.,

$$W : \mathsf{C} \times \mathsf{D} \to \mathsf{Vect}.$$

The (C, D)-bimodules form a functor category (C, D)-BiMod as usual. In particular, given any object $d \in \mathrm{ob}(\mathsf{D})$ we have that,

$$W(\bullet, d) \in \mathsf{C}\text{-}\mathsf{Mod}$$

sending $c \mapsto W(c, d)$. Similarly, for any $c \in \mathrm{ob}(\mathsf{C})$ we have that $W(c, \bullet) \in \mathsf{D}$-Mod. For simplicity, let C-BiMod denote the category of $(\mathsf{C}, \mathsf{C}^{\mathsf{op}})$-bimodules.

Example 2.2.6. Consider the category $(\mathsf{G}, \mathsf{G}^{\mathsf{op}})$-$\mathsf{BiMod}$ of bimodules over the product category $\mathsf{G} \times \mathsf{G}^{\mathsf{op}}$. Such a bimodule $W \in (\mathsf{G}, \mathsf{G}^{\mathsf{op}})$-$\mathsf{BiMod}$ corresponds precisely to a G-bimodule. Indeed, let $\bullet$ denote the unique object in the category G, and $\circ$ the unique object in the category G^{op}. Then the vector space $W(\bullet, \circ) \in \mathsf{Vect}$ admits the structure of a G-bimodule, with left action $g \cdot w := W(g, \circ)(w)$, and right action $w \cdot h := W(\bullet, h)(w)$ for all $g, h \in G$ and $w \in W(\bullet, \circ)$. Finally, the compatibility of these actions is equivalent to the commutativity of

$$\begin{tikzcd}
W(\bullet, \circ) \arrow[r, "W(g,\circ)"] \arrow[d, "W(\bullet,h)"'] & W(\bullet, \circ) \arrow[d, "W(\bullet,h)"] \\
W(\bullet, \circ) \arrow[r, "W(g,\circ)"] & W(\bullet, \circ)
\end{tikzcd}$$

which follows from the functoriality of W. Moreover, it is straightforward to see that the G-bimodule structure of $W(\bullet, \circ)$ completely determines the functor W.

2.2.2 Useful notions for representations of categories

Any category of functors from a small category C to Vect (or indeed, to any abelian category) is abelian [33]. In particular, the representation category C-Mod pointwise inherits the notions of submodule (or subrepresentation), direct sum, (co)kernel, injection, surjection and quotient from the parallel notions in Vect. So, for example, the C-module W is said to be a submodule of the C-module V if $W(A)$ is a subspace of $V(A)$ for all objects $A \in \mathrm{ob}(\mathsf{C})$, and any functor $W(A) \xrightarrow{W(f)} W(B)$ for $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ is obtained by restriction on $V(f)$. For another example, we write, for $V \in \mathsf{C}$-Mod,

$$V = V_1 \oplus V_2,$$

if $V(A) = V_1(A) \oplus V_2(A)$ for all $A \in \mathrm{ob}(\mathsf{C})$ where $V_1, V_2 \in \mathsf{C}$-Mod.

(Bi)Degree of a C-module. Let C be a category whose objects are finite sets. Then the category of representations C-Mod has a notion of degree.

Definition 2.2.7. (Degree) Let C be a category whose objects are finite sets and let $V \in \mathsf{C}$-Mod. For $a \in \mathbb{N}$, the degree a submodule $\deg_a(V) \subseteq V$ is the C-module defined by,

$$\deg_a(V)(S) = \begin{cases} V(S) & |S| = a \\ 0 & \text{else} \end{cases}$$

and we have that,

$$V = \bigoplus_{a \geq 0} \deg_a(V).$$

We say that V is supported in degree a if $V = \deg_a(V)$.

Similarly, let D be a category whose objects are pairs of finite sets. Then $\mathsf{D}\text{-}\mathsf{Mod}$ has a notion of bidegree.

Definition 2.2.8. (Bidegree) Let D be a category whose objects are pairs of finite sets and let $V \in \mathsf{D}\text{-}\mathsf{Mod}$. For $(a, b) \in \mathbb{N} \times \mathbf{N}$, the bi-degree (a, b) submodule $\deg_{a,b}(V) \subseteq V$ is the D-module defined by,

$$\deg_{a,b}(V)(S,T) = \begin{cases} V(S,T) & |S| = a, |T| = b \\ 0 & \text{else} \end{cases}$$

and we have that,

$$V = \bigoplus_{a,b \geq 0} \deg_{a,b}(V).$$

We say that V is supported in bidegree (a, b) if $V = \deg_{a,b}(V)$.

2.2.3 Representable functors

An important example of a bimodule over a product category is the representable functor R_{C}, the appropriate setting for which is *locally small* categories. That is, categories C for which $\mathrm{Hom}_{\mathsf{C}}(A, B)$ is a set for all $A, B \in \mathrm{ob}(\mathsf{C})$.

Definition 2.2.9. (The representable functor R_{C}.) Let C be any locally small category.

- Fix an object $A \in \mathrm{ob}(\mathsf{C})$ and define a functor,

  $$\mathrm{Hom}_{\mathsf{C}}(A, \bullet) : \mathsf{C} \to \mathsf{Set},$$

  sending $B \mapsto \mathrm{Hom}_{\mathsf{C}}(A, B)$ and sending the morphism $g \in \mathrm{Hom}_{\mathsf{C}}(B, B')$ to the map,

  $$\mathrm{Hom}_{\mathsf{C}}(A, B) \to \mathrm{Hom}_{\mathsf{C}}(A, B'),$$

  obtained by post-composition with g. Its linearization is a C-module $\Bbbk[\mathrm{Hom}_{\mathsf{C}}(A, \bullet)]$.

- Similarly, fix an object $B \in \text{ob}(\mathsf{C})$ and define a functor,

  $$\text{Hom}_{\mathsf{C}}(\bullet, B) : \mathsf{C}^{\mathsf{op}} \to \mathsf{Set},$$

  sending $A \mapsto \text{Hom}_{\mathsf{C}}(A, B)$ and sending the morphism $f \in \text{Hom}_{\mathsf{C}^{\mathsf{op}}}(A, A')$ to the map,

  $$\text{Hom}_{\mathsf{C}}(A, B) \to \text{Hom}_{\mathsf{C}}(A', B),$$

  obtained by pre-composition with f. Its linearization is a C^{op}-module $\Bbbk[\text{Hom}_{\mathsf{C}}(\bullet, B)]$.

- These functors are compatible in the sense that they extend to a bifunctor,

  $$\text{Hom}_{\mathsf{C}}(\bullet, \bullet) : \mathsf{C}^{\mathsf{op}} \times \mathsf{C} \to \mathsf{Set},$$

  sending $(A, B) \in \text{ob}(\mathsf{C}^{\mathsf{op}} \times \mathsf{C})$ to $\text{Hom}_{\mathsf{C}}(A, B)$. Morphisms in the product category $\mathsf{C}^{\mathsf{op}} \times \mathsf{C}$ are of the form (f, g) where $f \in \text{Hom}_{\mathsf{C}^{\mathsf{op}}}(A, A')$ is a morphism in C^{op} and $g \in \text{Hom}_{\mathsf{C}}(B, B')$ is a morphism in C. The morphism (f, g) is sent to the map,

  $$\text{Hom}_{\mathsf{C}}(A', B) \to \text{Hom}_{\mathsf{C}}(A, B'),$$

  obtained by simultaneously pre-composing with f and post-composing with g. Its linearization is the representable functor $\mathsf{R}_{\mathsf{C}}(\bullet, \bullet) := \Bbbk[\text{Hom}_{\mathsf{C}}(\bullet, \bullet)]$,

  $$\mathsf{R}_{\mathsf{C}}(\bullet, \bullet) : \mathsf{C}^{\mathsf{op}} \times \mathsf{C} \to \mathsf{Vect}$$

  sending the pair (A, B) to the vector space $\Bbbk[\text{Hom}_{\mathsf{C}}(A, B)]$.

Example 2.2.10. (The group algebra as the representable functor R_{G}.)
Consider a group G and its corresponding category G. The category G only has a single object, which we temporarily denote $\circ$. The representable functor R_{G} is given as,

$$\mathsf{R}_{\mathsf{G}}(\circ, \circ) = \Bbbk[\text{Hom}_{\mathsf{G}}(\circ, \circ)] = \Bbbk[G].$$

That is, the representable functor R_{G} is precisely the group algebra with its corresponding bimodule structure.

Remark 2.2.11. Following Definition 2.2.9, we have that $\mathsf{R}_\mathsf{G}(\bullet,\bullet) \in (\mathsf{G}^{\mathsf{op}},\mathsf{G})\text{-}\mathsf{BiMod}$. For notational consistency, it is preferable to consider the group algebra a $(\mathsf{G},\mathsf{G}^{\mathsf{op}})$-bimodule. This is done by applying the representable functor construction to the opposite category G^{op},

$$\mathsf{R}_{\mathsf{G}^{\mathsf{op}}}(\bullet,\bullet) \in (\mathsf{G},\mathsf{G}^{\mathsf{op}})\text{-}\mathsf{BiMod}.$$

We are now able to state the Yoneda lemma.

Theorem 2.2.12 (Yoneda lemma.)**.** *Let* C *be a (locally small) category and let* $A \in \mathrm{ob}(\mathsf{C})$*. Then for any functor* $F : \mathsf{A} \to \mathsf{Set}$*, there is a one-to-one correspondence between the set of natural transformations* $\mathrm{Nat}(\mathrm{Hom}_\mathsf{C}(A,\bullet),F)$ *and the set* $F(A)$*.*

Example 2.2.10, cont. Let $V \in \mathsf{G}\text{-}\mathsf{Mod}$, and let $\circ$ be the unique object in G. In this setting, natural transformations correspond to G-module homomorphisms, and thus the Yoneda lemma states that the G-module homomorphisms $\Bbbk[G] \to V$ are in bijection with V. Indeed, any G-module homomorphism $\phi : \Bbbk[G] \to V$ is determines and is determined by the image of the identity $\phi(1)$ in V.

2.2.4 Tensor product over a category

Recall the definition of tensor product over a category (e.g., [24]).

Definition 2.2.13. (Tensor product over C.) Let $V \in \mathsf{C}\text{-}\mathsf{Mod}$ and $W \in \mathsf{C}^{\mathsf{op}}\text{-}\mathsf{Mod}$. The tensor product $V \otimes_\mathsf{C} W \in \mathsf{Vect}$ can be defined as the coend,

$$\int^{c\in\mathsf{C}} V(c) \otimes W(c).$$

Concretely, this is the largest quotient of

$$\bigoplus_{c\in\mathrm{ob}(\mathsf{C})} V(c) \otimes W(c)$$

in which

$$v_c \otimes f^*(w_{c'}) \in V(c) \otimes W(c) \quad \text{is identified with} \quad f_*(v_c) \otimes w_{c'} \in V(c') \otimes W(c')$$

for all $c, c' \in \mathrm{ob}(\mathsf{C})$, $f \in \mathrm{Hom}_{\mathsf{C}}(c, c')$, and all $v_c \in V(c), w_{c'} \in W(c')$.

This construction is entirely functorial, and thus we have defined a bifunctor,

$$- \otimes_{\mathsf{C}} - : \mathsf{C}\text{-}\mathsf{Mod} \times \mathsf{C}^{\mathsf{op}}\text{-}\mathsf{Mod} \to \mathsf{Vect}.$$

Symmetrically we have a bifunctor,

$$- \otimes_{\mathsf{C}} - : \mathsf{C}^{\mathsf{op}}\text{-}\mathsf{Mod} \times \mathsf{C}\text{-}\mathsf{Mod} \to \mathsf{Vect}.$$

Example 2.2.14. Consider a group G and its corresponding category G. Let $V \in \mathsf{G}^{\mathsf{op}}\text{-}\mathsf{Mod}$ and $W \in \mathsf{G}\text{-}\mathsf{Mod}$. Then $V \otimes_{\mathsf{G}} W$ is the largest quotient of $V(\bullet) \otimes W(\bullet) = V \otimes W$ in which,

$$v \cdot g \otimes w \in V \otimes W \quad \text{is identified with} \quad v \otimes g \cdot w \in V \otimes W$$

for all $v \in V, w \in W$ and $g \in G$. This coincides with the definition of the tensor product $V \otimes_G W$ of a right G-module V with a left G-module W.

Properties of the tensor product. We record here some useful properties that the tensor product enjoys. A useful observation is that this categorical tensor product respects direct sums. This is an example of the well-known fact that coends commute with colimits (see, e.g., [26]).

Lemma 2.2.15. *Let* C *be a small category and let* $V_1, V_2, W \in \mathsf{C}\text{-}\mathsf{Mod}$*. Then,*

$$(V_1 \oplus V_2) \otimes_{\mathsf{C}} W \cong (V_1 \otimes_{\mathsf{C}} W) \oplus (V_2 \otimes_{\mathsf{C}} W)$$

We continue the analogy with R-modules, where we have the standard result that the tensor product over R of a right R-module V with an (R, S)-bimodule W is a right S-module $V \otimes_R W$. This generalizes to our setting as follows.

Lemma 2.2.16. *Let $V \in \mathsf{C}\text{-}\mathsf{Mod}$ and let $W \in (\mathsf{C}^{\mathsf{op}}, \mathsf{D})\text{-}\mathsf{BiMod}$. Then there is a functor*

$$V \otimes_{\mathsf{C}} W : \mathsf{D} \to \mathsf{Vect}$$

sending $d \in \mathrm{ob}(\mathsf{D})$ to the vector space $V \otimes_{\mathsf{C}} W(\bullet, d)$. That is, W determines a functor,

$$\bullet \otimes_{\mathsf{C}} W : \mathsf{C}\text{-}\mathsf{Mod} \to \mathsf{D}\text{-}\mathsf{Mod}.$$

Proof. Everything in sight is functorial. $\square$

Example 2.2.17. The representable functors $\mathsf{R}_{\mathsf{C}} \in (\mathsf{C}, \mathsf{C}^{\mathsf{op}})\text{-}\mathsf{BiMod}$ naturally give rise to functors,

$$\mathsf{R}_{\mathsf{C}} \otimes_{\mathsf{C}} \bullet : \mathsf{C}\text{-}\mathsf{Mod} \to \mathsf{C}\text{-}\mathsf{Mod}.$$

Dual to the categorical tensor product is the categorical Hom-functor. Concretely, given $W \in \mathsf{D}^{\mathsf{op}}\text{-}\mathsf{Mod}$ and $U \in \mathsf{D}\text{-}\mathsf{Mod}$, then $\mathrm{Hom}_{\mathsf{D}}(W, U)$ can be defined as the end,

$$\int_{d \in \mathsf{D}} \mathrm{Hom}(W(d), U(d)).$$

Similarly to above, given a bimodule $W \in (\mathsf{C}^{\mathsf{op}}, \mathsf{D})\text{-}\mathsf{BiMod}$ this can be extended to a functor,

$$\mathrm{Hom}_{\mathsf{D}}(W, \bullet) : \mathsf{D}\text{-}\mathsf{Mod} \to \mathsf{C}\text{-}\mathsf{Mod}.$$

and there is an adjunction,

$$\bullet \otimes_{\mathsf{C}} W : \mathsf{C}\text{-}\mathsf{Mod} \leftrightharpoons \mathsf{D}\text{-}\mathsf{Mod} : \mathrm{Hom}_{\mathsf{D}}(W, \bullet)$$

called the tensor-hom adjunction.

2.2.5 Frobenius reciprocity, take two

We restate the Frobenius reciprocity theorem in the language of categories. Recall the setting was a subgroup H of a group G. This translates to considering a subcategory H of a category G. We first reinterpret the operations of restriction and induction in this setting. It is easy to see that restriction corresponds to the functor,

$$\mathrm{Res}_{\mathsf{H}}^{\mathsf{G}}(\bullet) : \mathsf{G}\text{-}\mathsf{Mod} \to \mathsf{H}\text{-}\mathsf{Mod},$$

obtained by precomposition with the inclusion of categories functor $\mathsf{H} \hookrightarrow \mathsf{G}$. In the setting of G-modules, the induction functor was defined as the tensor product over H with the group ring,

$$\Bbbk[G] \otimes_H -.$$

where $\Bbbk[G]$ was interpreted as a (G, H)-bimodule. This corresponds to taking the tensor product over the category H with the representable functor $\mathsf{R}_{\mathsf{G}^{\mathsf{op}}}(\bullet, \bullet)$ considered as a $(\mathsf{G}, \mathsf{H}^{\mathsf{op}})$-bimodule, where the right action is obtained by restriction. By Lemma 2.2.16, we see that this determines the induction functor,

$$\mathrm{Ind}_{\mathsf{H}}^{\mathsf{G}}(\bullet) = \mathsf{R}_{\mathsf{G}^{\mathsf{op}}} \otimes_{\mathsf{H}} \bullet : \mathsf{H}\text{-}\mathsf{Mod} \to \mathsf{G}\text{-}\mathsf{Mod}.$$

Now the Frobenius reciprocity theorem manifests itself as a duality between the functors $\mathrm{Res}_{\mathsf{H}}^{\mathsf{G}}(\bullet)$ and $\mathrm{Ind}_{\mathsf{H}}^{\mathsf{G}}(\bullet)$.

Theorem 2.2.18 (The Frobenius reciprocity theorem.)**.** *There is an adjunction,*

$$\mathrm{Ind}_{\mathsf{H}}^{\mathsf{G}}(\bullet) : \mathsf{H}\text{-}\mathsf{Mod} \leftrightharpoons \mathsf{G}\text{-}\mathsf{Mod} : \mathrm{Res}_{\mathsf{H}}^{\mathsf{G}}(\bullet).$$

Proof. We need to show that there is a natural bijection,

$$\mathrm{Hom}_{\mathsf{G}}(\mathrm{Ind}_{\mathsf{H}}^{\mathsf{G}}(W), V) \cong \mathrm{Hom}_{\mathsf{H}}(W, \mathrm{Res}_{\mathsf{H}}^{\mathsf{G}}(V)).$$

This follows from the original statement of the theorem (Theorem 2.1.25) in light of Proposition 2.1.22. $\square$

2.2.6 The free object paradigm

One upshot of this reorganization into the language of categories is that paves a clear path for a generalization of the Frobenius reciprocity theorem to relate representations theories of various categories. Concretely, let C be a subcategory of a small category D. We start by defining restriction and induction in this setting.

Definition 2.2.19.

1. The restriction functor,

$$\mathrm{Res}^{\mathsf{D}}_{\mathsf{C}}(\bullet) : \mathsf{D}\text{-}\mathsf{Mod} \to \mathsf{C}\text{-}\mathsf{Mod},$$

   is defined by precomposition with the inclusion functor $\mathsf{C} \hookrightarrow \mathsf{D}$. In particular, it takes the D-module V to the C-module defined as the composition,

$$\begin{tikzcd} \mathsf{C} \arrow[rr, dashed, "\mathrm{Res}^{\mathsf{D}}_{\mathsf{C}}(V)"] \arrow[dr, hook] & & \mathsf{Vect} \\ & \mathsf{D} \arrow[ur, "V"'] & \end{tikzcd}$$

2. Let $\mathsf{R}_{\mathsf{D}}(\bullet)$ be the representable functor in the category D, considered as a $(\mathsf{C}^{\mathsf{op}}, \mathsf{D})$-bimodule. Define the induction functor $\mathrm{Ind}^{\mathsf{D}}_{\mathsf{C}}(\bullet)$ as the tensor product over C,

$$\mathrm{Ind}^{\mathsf{D}}_{\mathsf{C}}(\bullet) = \bullet \otimes_{\mathsf{C}} \mathsf{R}_{\mathsf{D}} : \mathsf{C}\text{-}\mathsf{Mod} \to \mathsf{D}\text{-}\mathsf{Mod}.$$

The analogue of the Frobenius reciprocity theorem is the statement that these functors form an adjoint pair.

Theorem 2.2.20. *Let $\mathsf{C} \subseteq \mathsf{D}$ as above. There is an adjunction,*

$$\mathrm{Ind}^{\mathsf{D}}_{\mathsf{C}}(\bullet) : \mathsf{C}\text{-}\mathsf{Mod} \leftrightharpoons \mathsf{D}\text{-}\mathsf{Mod} : \mathrm{Res}^{\mathsf{D}}_{\mathsf{C}}(\bullet).$$

Proof. We need to show that there is a natural bijection,

$$\mathrm{Hom}_{\mathsf{D}\text{-}\mathsf{Mod}}(\mathrm{Ind}^{\mathsf{D}}_{\mathsf{C}}(W), V) \cong \mathrm{Hom}_{\mathsf{C}\text{-}\mathsf{Mod}}(W, \mathrm{Res}^{\mathsf{D}}_{\mathsf{C}}(V)).$$

Applying the tensor-hom adjunction to the LHS we have,

$$\begin{aligned}\mathrm{Hom}_{\mathsf{D}\text{-}\mathsf{Mod}}(\mathrm{Ind}_{\mathsf{C}}^{\mathsf{D}}(W), V) &= \mathrm{Hom}_{\mathsf{D}\text{-}\mathsf{Mod}}(W \otimes_{\mathsf{C}} \mathsf{R}_{\mathsf{D}}, V)\\ &\cong \mathrm{Hom}_{\mathsf{C}\text{-}\mathsf{Mod}}(W, \mathrm{Hom}_{\mathsf{D}\text{-}\mathsf{Mod}}(\mathsf{R}_{\mathsf{D}}, V))\end{aligned}$$

Finally, we claim that the D-module homomorphisms $\mathrm{Hom}_{\mathsf{D}\text{-}\mathsf{Mod}}(\mathsf{R}_{\mathsf{D}}, V)$ coincide with the restriction $\mathrm{Res}_{\mathsf{C}}^{\mathsf{D}}(V)$ as desired. First, note that D-module homomorphisms $\mathrm{Hom}_{\mathsf{D}\text{-}\mathsf{Mod}}(\mathsf{R}_{\mathsf{D}}, V)$ are, by definition, the natural transformations, $\mathrm{Nat}(\mathsf{R}_{\mathsf{D}}, V)$. Further, note that this has the structure of a C-module, sending the object $A \in \mathrm{ob}(\mathsf{C})$ to

$$\mathrm{Nat}(\mathsf{R}_{\mathsf{D}}, V) \cong V(A),$$

where the isomorphism follows from the Yoneda Lemma (Theorem 2.2.12). It follows immediately that this C-module is exactly $\mathrm{Res}_{\mathsf{C}}^{\mathsf{D}}(V)$, as claimed. □

Remark 2.2.21. Informally, a free functor is one arising as a left adjoint to a forgetful functor. The restriction functors deserve to be called forgetful, and as such, their left adjoint, the induction functors, deserve to be called free.

Definition 2.2.22. Let $\mathsf{C} \subseteq \mathsf{D}$ as above, and let $W \in \mathsf{C}\text{-}\mathsf{Mod}$. We call the induced module,

$$\mathrm{Ind}_{\mathsf{C}}^{\mathsf{D}}(W) \in \mathsf{D}\text{-}\mathsf{Mod},$$

the free D-module on W relative to the inclusion $\mathsf{C} \hookrightarrow \mathsf{D}$. When the inclusion of categories is understood we will simply call these free D-modules.

We will turn to this construction of free D-modules in multiple settings. In the next section we will recall the theory of FI-modules in which this construction arises as a central construction. It will be useful to note that induction respects direct sums.

Lemma 2.2.23. *Let $\mathsf{C} \subseteq \mathsf{D}$ as above. Let $V, W \in \mathsf{C}\text{-}\mathsf{Mod}$. Then,*

$$\mathrm{Ind}_{\mathsf{C}}^{\mathsf{D}}(V \oplus W) = \mathrm{Ind}_{\mathsf{C}}^{\mathsf{D}}(V) \oplus \mathrm{Ind}_{\mathsf{C}}^{\mathsf{D}}(W).$$

Proof. This follows immediately from Lemma 2.2.15 since induction is defined as a tensor product. □

2.3 FI-modules

Sequences $\{V_n\}_{n\geq 0}$ of S_n-modules naturally arise in many places in mathematics, from combinatorics, to topology through to algebraic geometry and beyond. In [9] Church and Farb noticed that many of these sequences satisfy strong representation theoretic constraints governing, for example, their decomposition into irreducible S_n-modules and the growth of their dimension. These well-behaved sequences were said to satisfy *representation stability* (see Definition 2.3.7). Later, and together with Ellenberg and then with Nagpal ([7, 8]), they introduced a single notion encapsulating this phenomenon; a finitely generated FI-module. We recall the basic notions here.

Let FI be the category whose objects are finite sets and whose morphisms are given by injections. Concretely, $\mathrm{Hom}_{\mathsf{FI}}(S,T)$ is the set of injections from S into T. Recall that we denote by $\mathbf{n}$ the set $\{1,\dots,n\}$, and that the full subcategory FI' with objects $\mathbf{n}$ is a skeleton of FI. By convention, if $n=0$ let $\mathbf{n}=\varnothing$.

Definition 2.3.1. An FI-*module* V is a covariant functor from FI to Vect.

We follow the standard notation of [7]. For simplicity we denote the vector space $V(S)$ by V_S and the vector space $V(\mathbf{n})$ by V_n. If $f\in\mathrm{Hom}_{\mathsf{FI}}(S,T)$ then we denote the linear map $V(f): V_S\to V_T$ simply by f_*.

Remark 2.3.2. The advantages of this perspective are numerous. On the one hand, bundling together all of the data in the sequence $\{V_n\}_{n\geq 0}$ into a single object V serves as a conceptual simplification. Additionally, in only considering the linear map ϕ_n, consistent sequences missed most of the maps $V_n \to V_{n+1}$ implicit in the FI-module structure. Lastly, this framework of functors from small categories similar to FI into Vect is rich and has many applications outside of representation stable sequences. For a more comprehensive discussion on representation stability see [9].

It will be instructive in what follows to have a toy example to hand.

Example 2.3.3. (Toy example.) Let V be the FI-module sending the finite set S to $\mathbb{Q}^S$, the free $\mathbb{Q}$-vector space on S. Denote the basis element corresponding to $s \in S$ by e_s. A morphism $f \in \mathrm{Hom}_{\mathsf{FI}}(S, T)$ sends a basis element e_s in V_S to the basis element $e_{f(s)}$ in $\mathbb{Q}^T$. Denote this FI-module by $\mathbb{Q}^\bullet$.

The fundamental notion in the theory of FI-modules is that of *finite generation*.

Definition 2.3.4. (Span.) Let $V \in \mathsf{FI}\text{-}\mathsf{Mod}$ and let $S \subset \bigsqcup_i V_i$. Let $\mathrm{Span}_V(S)$ denote the minimal sub-FI-module of V containing S.

Definition 2.3.5. (Finite generation.) We say V is *generated by* S if $V = \mathrm{Span}_V(S)$. We say V is *finitely generated* if it is generated by a finite set S, and we say it is *generated in degree* $\leq d$ if it is generated by a set $S \subset \bigsqcup_{i=0}^d V_i$.

Example 2.3.6. (Toy example, cont.) The FI-module $V = \mathbb{Q}^\bullet$ above is finitely generated. To see this, let $S = \{e_1\} \subset V_1 = \mathbb{Q}^{\{1\}}$. Any sub-$\mathsf{FI}$-module W of V containing e_1 also contains $f_*(e_1) = e_{f(1)} \in V_S$ for every $f \in \mathrm{Hom}_{\mathsf{FI}}(\{1\}, S)$. It therefore contains every basis element e_s in $\mathbb{Q}^S$ for every finite sets S, and therefore contains V as a sub-FI-module. In short, we have $W = V$ as FI-modules.

2.3.1 Representation stability

An important feature of FI-modules is that they give rise to sequences of S_n-modules. Indeed, let $V \in \mathsf{FI\text{-}Mod}$ and consider the sequence $\{V_n\}_{n\geq 0}$. Since endomorphisms in FI of $\mathbf{n}$ are naturally isomorphic to the symmetric group S_n, each vector space V_n inherits the structure of a S_n-module. Moreover, the natural inclusions,

$$i_n : \{1, \ldots, n\} \hookrightarrow \{1, \ldots, n, n+1\},$$

(defined by $i_n(j) = j$) give rise to linear maps $\phi_n = V(i_n) : V_n \to V_{n+1}$ for all n. Functoriality implies that the following diagram commutes,

$$\begin{tikzcd} V_n \arrow[r, "\phi_n"] \arrow[d, "\sigma"] & V_{n+1} \arrow[d, "\sigma"] \\ V_n \arrow[r, "\phi_n"] & V_{n+1} \end{tikzcd}$$

for all $n \geq 0$ and all $\sigma \in \mathrm{S}_n$. Here σ acts on V_{n+1} by its image under the standard inclusion $\mathrm{S}_n \hookrightarrow \mathrm{S}_{n+1}$. In [9] such sequences were called *consistent*, and it was in this setting that representation stability was defined. Concretely, Maschke's theorem tells us that each V_n decomposes into irreducible S_n-modules P_λ for various partitions $\lambda \vdash n$. Church and Farb's key insight was that in many important examples the irreducible representations that appeared stabilised in a sense that we now make precise.

Definition 2.3.7. ([9], Definition 1.1) Let $\{V_n\}_{n\geq 0}$ be a consistent sequence of S_n-modules. The sequence is *representation stable* if, for n sufficiently large, the following three conditions hold.

1. (Injectivity) The maps ϕ_n are injective.
2. (Surjectivity) The span of the S_{n+1}-orbit of $\phi(V_n) \subseteq V_{n+1}$ is all of V_{n+1}.
3. (Multiplicities) The S_n-module V_n admits a decomposition into irreducible S_n-modules,

$$V_n \cong \bigoplus c_{\lambda,n} P(\lambda)_n, \tag{2.5}$$

with $0 \leq c_{\lambda,n} \leq \infty$. For each λ, the coefficient $c_{\lambda,n}$ is eventually independent of n.

The sequence is called **uniformly representation stable** with stable range $n \geq N$ if in addition, the multiplicities $c_{\lambda,n}$ are independent of n for all $n \geq N$ with no dependence on λ.

Example 2.3.8. (Toy example, cont.) Returning to our toy example, it is not hard to see that the maps $\phi_n : V_n \to V_{n+1}$ defined above satisfy the first two conditions. Further, we have the well-known decomposition into irreducible S_n-modules,

$$V_n = \mathbb{Q}^{\mathbf{n}} = \{a(e_1 + \cdots + e_n) : a \in \mathbb{Q}\} \oplus \{a_1 \cdot e_1 + \cdots + a_n \cdot e_n : \sum a_i = 0\}$$

The summand on the left is the trivial representation of S_n, namely $P_{(n)} = P(\varnothing)_n$. What's left is the standard representation of S_n, and corresponds to the partition $(n-1,1)$. Thus we have, for all $n \geq 2$, the decomposition,

$$\mathbb{Q}^n \cong P(\varnothing)_n \oplus P(\square)_n,$$

and the sequence $\{\mathbb{Q}^n\}_{n\geq 0}$ satisfies uniform representation stability with stable range $n \geq 2$.

In [7] representation stability was recovered as a finite generation property of FI-modules.

Theorem 2.3.9 ([7], Theorem 1.13)**.** *Let $V \in \mathsf{FI}\text{-}\mathsf{Mod}$. Then V is finitely generated if and only if $\{V_n\}_{n\geq 0}$ is representation stable and each V_n is of finite dimension.*

2.3.2 Stability degree and weight of an FI-module

There are two useful properties of an FI-module, stability degree and weight, that can be seen to control the underlying representation theory. Let V_s be an S_s-module. Recall that we denote by $(V_s)_{\mathrm{S}_s}$ the S_s-coinvariant quotient $V_s \otimes_{\Bbbk \mathrm{S}_s} \Bbbk$.

Definition 2.3.10. An FI-module V has stability degree t if for all $a \geq 0$, the maps $(V_{s+a})_{\mathrm{S}_s} \to (V_{s+1+a})_{\mathrm{S}_{s+1}}$ induced by the standard inclusions $I_s : \{1, \ldots, s\} \to \{1, \ldots, s+1\}$, are isomorphisms for all $s \geq t$.

We say V has injectivity degree $\mathcal{I}$ (resp. surjectivity degree $\mathcal{S}$) if the maps $(V_{s+a})_{\mathrm{S}_s} \to (V_{s+1+a})_{\mathrm{S}_{s+1}}$ are injective $\forall\ s \geq \mathcal{I}$ (resp. surjective $\forall\ s \geq \mathcal{S}$). We say V has stability type $(\mathcal{I}, \mathcal{S})$.

Definition 2.3.11. An FI-module V has weight $\leq d$ if for all $s \geq 0$ every irreducible component $P(\lambda)_s$ of V_s has $|\lambda| \leq d$.

Remark 2.3.12. A key property of weight is that it is preserved under subquotients and extensions. In fact, there is an alternate definition of weight: the collection of FI-modules over $\Bbbk$ of weight $\leq d$ is the minimal collection which contains all FI-modules generated in degree $\leq d$ and is closed under subquotients and extensions. For more details see [7].

Together, finite weight and stability of an FI-module V imply representation stability of the corresponding sequence of S_s-modules $\{V_s\}$. Moreover, the stability degree and weight give a measure of control on the representation stable range. The following result of Church-Ellenberg-Farb is the key to deducing representation stability of $\{H^i(\Gamma_{n,s})\}_{s \geq 0}$ from our FI-module $H^i(\Gamma_{n,\bullet})$.

Proposition 2.3.13 ([7], Proposition 3.3.3)**.** *Let V be an FI-module of weight d and stability degree t. Then the sequence $\{V_s\}$ is uniformly representation stable with stable range $s \geq t + d$.*

2.3.3 Free FI-modules

In the course of the next few sections we define two important functors in the theory of FI-modules. This material is by now well-established. We largely follow Church-Ellenberg-Farb's original paper [7], albeit with a slightly modified notation.

1. The free functor. We define,

$$\mathrm{Ind}_{\mathsf{FB}}^{\mathsf{FI}}(\bullet) : \mathsf{FB}\text{-}\mathsf{Mod} \to \mathsf{FI}\text{-}\mathsf{Mod},$$

   the left-adjoint to the restriction functor $\mathrm{Res}_{\mathsf{FB}}^{\mathsf{FI}} : \mathsf{FI}\text{-}\mathsf{Mod} \to \mathsf{FB}\text{-}\mathsf{Mod}$. We will realise this functor as a *tensor product over* FB following our free-object paradigm,

$$\mathrm{Ind}_{\mathsf{FB}}^{\mathsf{FI}}(\bullet) = \bullet \otimes_{\mathsf{FB}} \mathsf{R}_{\mathsf{FI}}.$$

2. The functor $H_0(\bullet)$. We define,

$$H_0(\bullet) : \mathsf{FI}\text{-}\mathsf{Mod} \to \mathsf{FB}\text{-}\mathsf{Mod},$$

   a left-adjoint to a certain functor $\zeta(\bullet) : \mathsf{FB}\text{-}\mathsf{Mod} \to \mathsf{FI}\text{-}\mathsf{Mod}$. We will realise this functor as a tensor product over FI,

$$H_0(\bullet) = \bullet \otimes_{\mathsf{FI}} K$$

   where K is a certain $(\mathsf{FI}^{\mathsf{op}}, \mathsf{FI})$-bimodule defined below.

We apply the free object paradigm described in Section 2.2.6. Concretely, consider the inclusion of categories,

$$\mathsf{FB} \hookrightarrow \mathsf{FI}.$$

This gives rise to two functors,

$$\mathrm{Res}_{\mathsf{FB}}^{\mathsf{FI}} : \mathsf{FI}\text{-}\mathsf{Mod} \to \mathsf{FB}\text{-}\mathsf{Mod}, \qquad \mathrm{Ind}_{\mathsf{FB}}^{\mathsf{FI}} : \mathsf{FB}\text{-}\mathsf{Mod} \to \mathsf{FI}\text{-}\mathsf{Mod},$$

which by Theorem 2.2.20 determines an adjunction,

$$\mathrm{Ind}_{\mathsf{FB}}^{\mathsf{FI}} : \mathsf{FB}\text{-}\mathsf{Mod} \leftrightharpoons \mathsf{FI}\text{-}\mathsf{Mod} : \mathrm{Res}_{\mathsf{FB}}^{\mathsf{FI}} .$$

Let $W \in \mathsf{FB}\text{-}\mathsf{Mod}$. In [7] they give an explicit description of $\mathrm{Ind}_{\mathsf{FB}}^{\mathsf{FI}}(W)$. Note that in [7] they refer to the functor $\mathrm{Ind}_{\mathsf{FB}}^{\mathsf{FI}}(\bullet)$ simply as M.

Proposition 2.3.14. *Let $W \in \mathsf{FB}\text{-}\mathsf{Mod}$. Then the FI-module $\mathrm{Ind}_{\mathsf{FB}}^{\mathsf{FI}}(W) = W \otimes_{\mathsf{FB}} \mathsf{R}_{\mathsf{FI}}$ sends the finite set S to the vector space,*

$$\bigoplus_{T \subseteq S} W_T.$$

Proof. We have that,

$$\mathrm{Ind}_{\mathsf{FB}}^{\mathsf{FI}}(W) = \bigoplus_{a \geq 0} \mathrm{Ind}_{\mathsf{FB}}^{\mathsf{FI}}(\deg_a W)$$

where W_a is the degree a submodule of W. Therefore, without loss of generality, assume that W is supported in degree a. Let $S \in \mathrm{ob}(\mathsf{FI})$. We have,

$$\left(\mathrm{Ind}_{\mathsf{FB}}^{\mathsf{FI}} W\right)_S = \bigoplus_{T : |T| = a} W_T \otimes_{\mathsf{FB}} \Bbbk[\mathrm{Hom}_{\mathsf{FI}}(T, S)] \cong W_T \otimes_{\mathrm{Sym}(T)} \Bbbk[\mathrm{Hom}_{\mathsf{FI}}(T, S)],$$

for a fixed set $T \in \mathrm{ob}(\mathsf{FI})$ of size a. Observe that we can write $\mathrm{Hom}_{\mathsf{FI}}(T, S)$ as a sum

$$\mathrm{Hom}_{\mathsf{FI}}(T, S) = \bigoplus_{\substack{I \subseteq S \\ |I| = |T|}} \mathrm{Hom}_{\mathsf{FI}}(T, S; I),$$

where $\mathrm{Hom}_{\mathsf{FI}}(T, S; I) \subseteq \mathrm{Hom}_{\mathsf{FI}}(T, S)$ consists of those injections $f : T \hookrightarrow S$ such that $\mathrm{im}(f) = I$. Note that $\mathrm{Hom}_{\mathsf{FI}}(T, S; I) \cong \mathrm{Sym}(T)$, and thus we have,

$$\left(\mathrm{Ind}_{\mathsf{FB}}^{\mathsf{FI}} W\right)_S = \bigoplus_{\substack{I \subseteq S \\ |I| = |T|}} W_T \otimes_{\mathrm{Sym}(T)} \Bbbk[\mathrm{Sym}(T)] \cong \bigoplus_{\substack{T \subseteq S \\ |T| = a}} W_T.$$

The result follows. $\square$

Yet another useful description of the free functors comes from making the same construction in the skeletal subcategories of $\mathsf{FB}' \subseteq \mathsf{FB}$ and $\mathsf{FI}' \subseteq \mathsf{FI}$. Recall, these are the full subcategories of FB and FI with objects the finite sets $\{\mathbf{n} : n \in \mathbb{N}\}$.

Proposition 2.3.15. *Let $W \in \mathsf{FB}'\text{-}\mathsf{Mod}$. Then the FI'-module $\mathrm{Ind}_{\mathsf{FB}'}^{\mathsf{FI}'}(W)$ satisfies,*

$$\mathrm{Ind}_{\mathsf{FB}'}^{\mathsf{FI}'}(W)_n \cong \bigoplus_{a \leq n} \mathrm{Ind}_{\mathrm{S}_a \times \mathrm{S}_{n-a}}^{\mathrm{S}_n} W_a \boxtimes \Bbbk.$$

Proof. Similarly to the proof above, we assume, without loss of generality, that W is supported in degree a, so that $W = W_a$. By definition,

$$\mathrm{Ind}_{\mathsf{FB}'}^{\mathsf{FI}'}(W_a) = W_a \otimes_{\mathsf{FB}'} \mathsf{R}_{\mathsf{FI}'} \cong W_a \otimes_{\mathrm{S}_a} \Bbbk[\mathrm{Hom}_{\mathsf{FI}'}(\mathbf{a}, \mathbf{n})].$$

Again, as above, observe that $\mathrm{Hom}_{\mathsf{FI}'}(\mathbf{a}, \mathbf{n})$ spits as a sum,

$$\mathrm{Hom}_{\mathsf{FI}'}(\mathbf{a}, \mathbf{n}) = \bigoplus_{\substack{I \subseteq \mathbf{n} \\ |I| = a}} \mathrm{Hom}_{\mathsf{FI}'}(\mathbf{a}, \mathbf{n}; I),$$

where $\mathrm{Hom}_{\mathsf{FI}'}(\mathbf{a}, \mathbf{n}; I) \subseteq \mathrm{Hom}_{\mathsf{FI}'}(\mathbf{a}, \mathbf{n})$ consists of those injections $f : \mathbf{a} \hookrightarrow \mathbf{n}$ with $\mathrm{im}(f) = I$. We have that $\mathrm{Hom}_{\mathsf{FI}'}(\mathbf{a}, \mathbf{n}; I)$ is isomorphic to S_a. Concretely, fix an identification of $\mathbf{a}$ with the subset $I \subseteq \mathbf{n}$. This gives the isomorphism between the bijections $\mathbf{a} \to I$ and S_a. Further, the complement of I in $\mathbf{n}$ identifies a subgroup of S_n isomorphic to S_{n-a} which acts trivially on S_a. Putting this together gives,

$$W_a \otimes_{\mathrm{S}_a} \Bbbk[\mathrm{Hom}_{\mathsf{FI}'}(\mathbf{a}, \mathbf{n}; I)] \cong W_a \otimes_{\mathrm{S}_a} \Bbbk[\mathrm{S}_a] \cong W_a$$

Furthermore, the subsets I are in bijection with the cosets $\mathcal{A} := \mathrm{S}_n \,/\, \mathrm{S}_a \times \mathrm{S}_{n-a}$ and we have,

$$\mathrm{Ind}_{\mathsf{FB}'}^{\mathsf{FI}'}(W_a) \cong \bigoplus_{\mathcal{A}} W_a.$$

Finally, observe that the stabilizer of W_a in S_n is $\mathrm{S}_a \times \mathrm{S}_{n-a}$, and the result follows from Lemma 2.1.15. □

2.3.4 Finite generation in FI-Mod.

The following is an important example of an induced module central to the notion of finitely generation in the category FI-Mod. It was originally introduced in [7], where they called it $M(m)$.

Definition 2.3.16. For any $m \geq 0$, let $M(m)$ denote the representable functor,

$$\mathsf{R}_{\mathsf{FI}}(\mathbf{m}, \bullet) = \Bbbk[\mathrm{Hom}_{\mathsf{FI}}(\mathbf{m}, \bullet)] \in \mathsf{FI}\text{-}\mathsf{Mod}.$$

It takes any finite set $S \in \mathrm{ob}(\mathsf{FI})$ to the vector space $\Bbbk[\mathrm{Hom}_{\mathsf{FI}}(\mathbf{m}, S)]$ and any morphism $f \in \mathrm{Hom}_{\mathsf{FI}}(S, S')$ to the map induced by post-composition with f. Notice that $M(m)$ is exactly the FI-module $\mathrm{Ind}_{\mathsf{FB}}^{\mathsf{FI}}(\Bbbk[\mathrm{S}_m])$, where here $\Bbbk[\mathrm{S}_m]$ is considered as an FB-module supported in degree m.

Remark 2.3.17. The Yoneda lemma implies that

$$\mathrm{Hom}_{\mathsf{FI}\text{-}\mathsf{Mod}}(M(m), V) \cong V_m.$$

In other words, the morphism of FI-modules $M(m) \to V$ is completely determined by the image $v_\star$ of $(id_m : \mathbf{m} \to \mathbf{m}) \in M(m)_m$ in V_m. In particular, notice that the image of $M(m)$ in V coincides with $\mathrm{Span}_V(v_\star)$. This observation motivates the following result of [7].

Lemma 2.3.18. *([7], Proposition 2.3.5) An FI-module V is finitely generated if and only if there exists a surjection*

$$\bigoplus_i M(m_i) \twoheadrightarrow V.$$

for some finite sequence of integers $\{m_i\}$.

Remark 2.3.19. An FI-module V is *free* if it is of the form $V = \bigoplus_i M(m_i)$. This is consistent with the notion of a finitely generated R module as one admitting a surjection from a free R-module.

2.3.5 The FI-modules $M(\lambda)$ and $P(\lambda)$

Fix a partition $\lambda = (\lambda_1, \lambda_2, \cdots) \vdash a$. Recall the padded partitions $\lambda[n]$ and their associated irreducible S_n-module $P(\lambda)_n$ from Definition 2.1.5. In [7] they prove that these determine a finitely generated FI-module. We briefly recall that construction.

Definition 2.3.20. Fix a partition λ. Denote the free FI-module $\mathrm{Ind}_{\mathsf{FB}}^{\mathsf{FI}}(P_\lambda)$ simply by $M(\lambda)$. Explicitly $M(\lambda)$ takes a finite set $\mathbf{s} = \{1, \ldots, s\}$ to $P_\lambda \circledast P_{(s-|\lambda|)}$.

Remark 2.3.21. This notation is consistent with the notation in [7].

In [7] they describe the stability degree and weight of $M(\lambda)$. We stress that we are working over a field $\Bbbk$ of characteristic zero, otherwise we cannot guarantee such strong bounds on surjectivity degree.

Proposition 2.3.22 (see [7])**.** *The FI-module $M(\lambda)$ has stability type $(0, \lambda_1)$ and weight at most $|\lambda|$.*

Proof. This follows from three results in [7]. The injectivity degree is given in Proposition 3.1.7, the surjectivity degree in Proposition 3.2.6 and the weight in Proposition 3.2.4. □

Lemma 2.3.23 ([7], Proposition 3.4.1)**.** *For any partition λ, there is a finitely generated FI-module $P(\lambda)$, obtained as a sub-FI-module of the free FI-module $M(\lambda)$, satisfying,*

$$P(\lambda)_n = \begin{cases} P_{\lambda[n]} & n \geq |\lambda| + \lambda_1 \\ 0 & \textit{else} \end{cases}$$

Remark 2.3.24.

1. This justifies the notation $P(\lambda)_n$ for irreducible S_n-module associated to the padded partition $\lambda[n]$.
2. The FI-module $P(\lambda)_n$ is readily seen to be an FI-submodule of $\mathrm{Ind}_{\mathsf{FB}}^{\mathsf{FI}}(P_\lambda)_n$.

2.3.6 Homological techniques for FI-modules

One advantage of the FI-modules viewpoint is that it brings homological techniques to bear. In particular, the following result governs the dynamics of stability type through our spectral sequence. Recall that $\Bbbk$ has characteristic 0.

Proposition 2.3.25 ([7], Lemma 6.3.2)**.**

1. *Let U, V, W be FI-modules with stability type $(*, A), (B, C), (D, *)$ respectively, and let*
$$U \xrightarrow{f} V \xrightarrow{g} W$$
be a complex of FI-modules (i.e., $g \circ f = 0$). Then $\ker g / \operatorname{im} f$ has injectivity degree $\leq \max(A, B)$ and surjectivity degree $\leq \max(C, D)$.

2. *Let V be an FI-module with a filtration*
$$0 = F_j V \subseteq F_{j-i} V \subseteq \cdots \subseteq F_1 V \subseteq F_0 V = V$$
by FI-modules $F_i V$. The successive quotients $F_i V / F_{i+1} V$ have stability type $(\mathcal{I}, \mathcal{S})$ for all i if and only if V has stability type $(\mathcal{I}, \mathcal{S})$.

The functor H_0^{FI}. We have analyzed the inclusion $\mathsf{FB} \hookrightarrow \mathsf{FI}$ and its associated adjunction,
$$\operatorname{Ind}_{\mathsf{FB}}^{\mathsf{FI}}(\bullet) : \mathsf{FB}\text{-}\mathsf{Mod} \leftrightharpoons \mathsf{FI}\text{-}\mathsf{Mod} : \operatorname{Res}_{\mathsf{FB}}^{\mathsf{FI}} .$$
Consider now the inclusion of categories,
$$\zeta(\bullet) : \mathsf{FB}\text{-}\mathsf{Mod} \to \mathsf{FI}\text{-}\mathsf{Mod}$$
that extends an FB-module W to an FI-module $\zeta(W)$ by declaring the image of any non-bijective injection $f \in \operatorname{Hom}_{\mathsf{FI}}(S, T)$ to be the zero map in $\zeta(W)$. In [7] the define a left-adjoint to $\zeta(\bullet)$. We recall that construction now (also following notions from [6]).

Definition 2.3.26. Define a bimodule $K \in (\mathsf{FB}^{\mathsf{op}}, \mathsf{FB})\text{-}\mathsf{BiMod}$ as follows. Let $(S, T) \in \operatorname{ob}(\mathsf{FI}^{\mathsf{op}} \times \mathsf{FI})$ be a pair of finite sets and declare,
$$K(S, T) = \Bbbk[\operatorname{Hom}_{\mathsf{FB}}(S, T)],$$

In particular, if $|S| \neq |T|$ then $K(S,T) = 0$. Morphisms are of the form (f,g) where $f \in \mathrm{Hom}_{\mathsf{FB}^{\mathsf{op}}}(S,S')$ and $g \in \mathrm{Hom}_{\mathsf{FB}}(T,T')$. Declare that $(f,g)_*$ acts simultaniously by pre-composition with f and post-composition with g.

We promote K to an $(\mathsf{FI}^{\mathsf{op}}, \mathsf{FI})$-bimodule by declaring any non-bijective morphisms to induce the zero map.

Definition 2.3.27. Define the functor $H_0(\bullet)$ as the tensor product over FI with K considered as a $(\mathsf{FI}^{\mathsf{op}}, \mathsf{FI})$-bimodule,

$$\bullet \otimes_{\mathsf{FI}} K : \mathsf{FI}\text{-}\mathsf{Mod} \to \mathsf{FB}\text{-}\mathsf{Mod},$$

We extend this to a functor $H_0^{\mathsf{FI}} : \mathsf{FI}\text{-}\mathsf{Mod} \to \mathsf{FI}\text{-}\mathsf{Mod}$ defined as the composition $\zeta \circ H_0$

Lemma 2.3.28 ([6])**.** *The functor $H_0(\bullet)$ is the left-adjoint to $\zeta(\bullet)$. Explicitly, it satisfies*

$$H_0(V)_S = V_S / (\mathrm{Span}_V(V_{<S})_S). \tag{2.6}$$

Remark 2.3.29. The functor H_0^{FI} captures the notion of minimal generators of an FI-module. In particular, comparing (2.6) with Definition 2.3.5 gives:

1. An FI-module V is finitely generated if and only if the vector space,

$$\bigoplus_{n \geq 0} H_0^{\mathsf{FI}}(V)_n,$$

is finitely dimensional.

2. An FI-module V is generated in degree $\leq d$ if and only if,

$$H_0^{\mathsf{FI}}(V)_S = 0,$$

for all $|S| > d$.

2.3.7 Character Polynomials

For $j \geq 1$, let $X_j : \mathrm{S}_s \to \mathbb{N}$ be the class function defined by

$$X_j(\sigma) = \text{number of } j\text{-cycles in } \sigma.$$

A polynomial in the variables X_j is called a character polynomial. We define the degree of a character polynomial by setting $\deg(X_j) = j$. The following theorem of Church-Ellenberg-Farb says that characters of finitely generated FI-modules are eventually described by a single character polynomial, and moreover gives explicit bounds on the degree and the stable range of this polynomial in terms of weight and stability degree of the FI-module.

Theorem 2.3.30 ([7], Theorem 3.3.4)**.** *Let V be a finitely generated FI-module of weight $\leq d$ and stability degree $\leq t$. There exists a unique polynomial $f_V \in \mathbb{Q}[X_1, \ldots, X_d]$ of degree at most d such that for all $n \geq d + t$ and all $\sigma \in \mathrm{S}_n$,*

$$\chi_{V_n}(\sigma) = f_V(\sigma).$$

CHAPTER 3

ON THE FI-MODULE STRUCTURE OF $H^i(\Gamma_{n,s})$

It is well known that the group of outer automorphisms of the free group of rank n can be described as the space of self-homotopy equivalences of a graph X_n of rank n, up to homotopy, i.e.,

$$\mathrm{Out}(F_n) \cong \pi_0(HE(X_n)).$$

Similarly the full group of automorphisms of the free group of rank n is the space of homotopy equivalences of a graph $X_{n,1}$ of rank n with a distinguished basepoint ∂, up to homotopy,

$$\mathrm{Aut}(F_n) \cong \pi_0(HE(X_{n,1})),$$

where homotopies are required to fix the basepoint throughout.

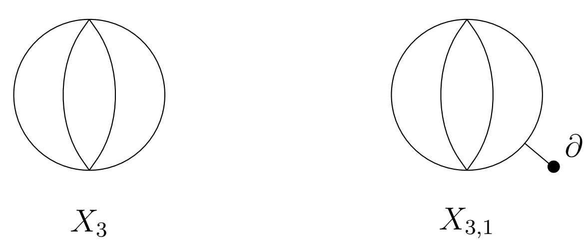


Figure 3.1: Examples of rank 3 graphs that can be used to define $\mathrm{Out}(F_3)$ and $\mathrm{Aut}(F_3)$.

There is a natural generalisation then, where we let $X_{n,s}$ be a graph, by which we mean a connected finite 1-dimensional CW-complex, of rank n with s marked points $\partial = \{x_1, \cdots, x_s\}$. We should then consider the group of self-homotopy equivalences of $X_{n,s}$ fixing ∂ pointwise, modulo homotopies through such maps, i.e.,

$$\Gamma_{n,s} := \pi_0(HE(X_{n,s})).$$

3.1 The cohomology of $\Gamma_{n,s}$

In this section we study the structure of the cohomology $H^i(\Gamma_{n,s})$, always over a field of characteristic zero, as a sequence of S_s-modules. The symmetric group S_s acts on $H^i(\Gamma_{n,s})$ as follows. A homotopy equivalence $h : X_{n,s} \to X_{n,s}$ permuting ∂ induces an automorphism of $\Gamma_{n,s}$ by conjugation. This automorphism depends, *a priori*, on the choice of h, however, on the level of cohomology it depends only on the permutation. Indeed, if h fixes ∂ pointwise then the induced automorphism is inner, and thus induces the identity on cohomology.

The groups $\Gamma_{n,s}$ have been used, for example, to show that $\mathrm{Out}(F_n)$ and $\mathrm{Aut}(F_n)$ satisfy homological stability in [16, 17], and they appeared in [3] in the proof that $\mathrm{Out}(F_n)$ is a virtual duality group. More recently they were used in [10] to investigate the so called unstable cohomology of $\mathrm{Out}(F_n)$ and $\mathrm{Aut}(F_n)$ by means of an 'assembly map'

$$H^i(\Gamma_{n_1,s_1}) \otimes \cdots \otimes H^i(\Gamma_{n_k,s_k}) \to H^i(\Gamma_{n,s}).$$

In particular, in [10] they compute $H^i(\Gamma_{n,s})$ as an S_s-module for rank $n = 1, 2$ and use these computations to assemble homology classes in the unstable range of $\mathrm{Out}(F_n)$ and $\mathrm{Aut}(F_n)$. Moreover these computations show that, in rank $n = 1, 2$ and for fixed $i \geq 0$ the sequence $\{H^i(\Gamma_{n,s})\}_{s\geq 0}$ satisfies representation stability (see Definition 2.3.7). In [10] they use an alternate description of $\Gamma_{n,s}$ as a quotient of a certain mapping class group of a three-manifold, together with general results about representation stability of mapping class groups, to deduce that for any fixed i and n the groups $H^i(\Gamma_{n,s})$ satisfy representation stability with stable range $s \geq 3i$. However, the calculations made in [10] in rank $n = 1, 2$ actually adhere to a bound of $s \geq i + n$. In this section we improve the stable range to agree with these low rank calculations.

Theorem 3.1.1. *For fixed i and n, the sequence,*

$$\{H^i(\Gamma_{n,s}) : s \in \mathbb{N}\},$$

is uniformly representation stable with stable range $s \geq n + i$.

We show this by exhibiting that $H^i(\Gamma_{n,s})$ defines an FI-module. Building on their work in [9], and together with Jordan Ellenberg and Rohit Nagpal, the theory of FI-modules was developed [7, 8], facilitating the application of homological techniques to sequences of S_s-modules. We use these techniques to prove the following theorem.

Theorem 3.1.2. *The FI-module $H^i(\Gamma_{n,\bullet})$ is finitely generated of stability degree n and weight i.*

An important feature of finitely generated FI-modules is the existence of character polynomials; integer-valued polynomials in $\mathbb{Q}[X_1, X_2, \ldots]$ where $X_i : \mathrm{S}_s \to \mathbb{N}$ is the class function that counts the number of i-cycles. Let $\chi_{H^i(\Gamma_{n,s})}$ denote the character of the S_s-module $H^i(\Gamma_{n,s})$.

Corollary 3.1.3. *There exists a character polynomial $f \in \mathbb{Q}[X_1, \ldots, X_i]$ depending on i and n such that for all $s \geq i + n$ and all $\sigma \in \mathrm{S}_s$,*

$$\chi_{H^i(\Gamma_{n,s})}(\sigma) = f(\sigma).$$

In particular, the dimension of $H^i(\Gamma_{n,s})$ is given by the polynomial $f(s, 0, \ldots, 0)$.

One consequence of this result is that, for s sufficiently large, the character $\chi_{i,n}$ is insensitive to cycles of length greater than i. We highlight this phenomenon by computing examples of these stable character polynomials in Section 2.3.7.

Theorem 3.1.1 and Corollary 3.1.3 follow immediately from Theorem 3.1.2 in light of Proposition 2.3.13 and Proposition 2.3.30.

Recall, for P an S_a-module, and Q an S_b-module, we denote the induced representation by

$$P \circledast Q := \mathrm{Ind}_{\mathrm{S}_a \times \mathrm{S}_b}^{\mathrm{S}_{a+b}} P \otimes Q$$

We denote by $V^{\wedge k}$ the S_k-module which is isomorphic as a vector space to $V^{\otimes k}$ where S_k acts by permuting the factors and multiplying by the sign of the permutation. That is,

$$V^{\wedge k} = V^{\otimes k} \otimes \epsilon_k.$$

With this in hand it is clear that Theorem 3.1.1 is an immediate corollary to Theorem 3.1.2. Another consequence of Theorem 3.1.2 is the existence of stable character polynomials. Corollary 3.1.3 will thus follow immediately from Theorem B. It is worth pointing out that in particular, this shows that the dimension of $H^i(\Gamma_{n,s})$ is eventually polynomial (as s grows but i and n remain fixed) given by a single character polynomial.

In [10] Conant-Hatcher-Kassabov-Vogtmann describe the S_s-module structure of $H^i(\Gamma_{n,s})$ for $n = 1, 2$, from which one can read off their irreducible S_s-module decomposition, that is, a decomposition into terms of the form $P(\lambda)_s$. We have the following classical fact, which underpins the theorem above. Fix a partition λ. There exists a unique character polynomial f_λ such that for any $s \geq |\lambda| + \lambda_1$, the character of the S_s-module $P(\lambda)_s$ is given by f_λ. In [14] they describe an algorithm constructing f_λ that we will use in conjunction with calculations from [10] to compute some explicit examples of character polynomials of various $H^i(\Gamma_{n,s})$. It is cleanest to describe these character polynomials in terms of the notation $(x)_j := x(x-1)\cdots(x-j+1)$.

Example. Fix $n = 1$ and $i = 2$. From [10], Proposition 2.7 we obtain the following decomposition of $H^2(\Gamma_{1,s})$ into irreducible S_s-modules.

$$H^2(\Gamma_{1,s}) = P\left(\ydiagram{1,1}\right)_s.$$

Using the algorithm from [14] we obtain the character polynomial $f_{\ydiagram{1,1}}$ for $P\left(\ydiagram{1,1}\right)$. Corollary 3.1.3 implies that, for $s \geq 3$, the character $\chi_{2,1}$ of $H^2(\Gamma_{1,s})$ is given by the polynomial,

$$f_{2,1}(X_1, X_2) = f_{\ydiagram{1,1}}(X_1, X_2) = \frac{1}{2} \cdot (X_1)_2 - (X_1) - (X_2) + 1.$$

We can use this, for example, to obtain that for $s \geq 3$ the dimension of $H^2(\Gamma_{1,s})$ is

$$\frac{s(s-1)}{2} - s + 1 = \binom{s-1}{2}.$$

Notice that this agrees with the description of $H^2(\Gamma_{1,s}) = \bigwedge^2 \Bbbk^{s-1}$ given in [10].

Example. Fix $n = 2$ and $i = 4$. From [10], Theorem 2.10 we obtain the following stable decomposition of $H^4(\Gamma_{2,s})$ into irreducible S_s-modules. For $s \geq 6$,

$$H^4(\Gamma_{2,s}) = P\left(\ydiagram{2}\right)_s \oplus P\left(\ydiagram{2,1}\right)_s \oplus P\left(\ydiagram{2,2}\right)_s.$$

Using the algorithm from [14] we obtain the character polynomials $f_{\ydiagram{2}}$, $f_{\ydiagram{2,1}}$ and $f_{\ydiagram{2,2}}$. Corollary 3.1.3 implies that, for $s \geq 6$, the character $\chi_{4,2}$ of $H^4(\Gamma_{2,s})$ is given by the sum of these three character polynomials,

$$f_{4,2}(X_1, X_2, X_3, X_4) = \frac{1}{12}(X_1)_4 + (X_2)_2 - X_1 \cdot X_3.$$

For instance, let $\tau = (1\ 2)(3\ 4)(5\ 6 \cdots 100) \in \mathrm{S}_{100}$. Then $\chi_{4,2}(\tau) = 2$.

Both the stable decomposition of $H^i(\Gamma_{n,s})$ and the stable character polynomials describing $\chi_{i,n}$ evident in these examples are general features of being a finitely generated

FI-module. It thus remains to prove Theorem 3.1.2, which we will do by analysing a spectral sequence of FI-modules. It turns out that the E_2-page of that spectral sequence admits a particularly nice description in terms of *free* FI-*modules*.

3.2 The FI-module structure

Let $X_{n,s}$ be a graph of rank n with s marked points $\partial := \{x_1, \ldots, x_s\}$. We defined $\Gamma_{n,s}$ as the space of self homotopy equivalences of $X_{n,s}$ fixing ∂ (pointwise) modulo homotopies that fix ∂ throughout. The group operation on $\Gamma_{n,s}$ is induced by composition of homotopy equivalences, which is clearly associative and admits an identity element. In [10] they prove the existence of inverse as follows. Let $f : X \to Y$ be a homotopy equivalence of graphs that sends $\partial_X = \{x_1, \ldots, x_s\} \subset X$ bijectively to $\partial_Y = \{y_1, \ldots, y_s\} \subset Y$. Consider the mapping cylinder of f, or rather its quotient Z obtained by collapsing the s intervals $x_i \times I$. By observing that the inclusion of Y into Z is a homotopy equivalence, and that Z deformation retracts onto X we obtain an inverse to f that acts as f^{-1} on ∂_Y as desired. Moreover, this argument shows that $\Gamma_{n,s}$ does not depend on the choice of graph $X_{n,s}$ up to isomorphism.

The proof of Theorem 3.1.2 in the case when $n > 1$ relies on a spectral sequence argument that itself is borne of certain short exact sequences we describe now (for full details see [10], Section 1.2).

Let $n > 1$, $s \geq 0$ and write $X = X_{n,s}$. Let E denote the space of homotopy equivalences of X with no requirement that ∂ be fixed, and let D be the space of homotopy equivalences of X that are required to fix ∂. Thus $\Gamma_{n,s} \cong \pi_0(D)$ and $\Gamma_{n,0} = \mathrm{Out}(F_n) \cong \pi_0(E)$. There is a map $E \to X^s$ by $f \mapsto (f(x_1), \ldots, f(x_s))$ which is a fibration with fiber D over the point $(x_1, \ldots, x_s)$. The long exact sequence of homotopy groups of this fibration ends,

$$\pi_1(E) \to \pi_1(X^s) \to \Gamma_{n,s} \to \mathrm{Out}(F_n) \to 1. \tag{3.1}$$

To see that $\pi_1(E) = 1$ we consider the case $s = 1$, when (3.1) says,

$$\pi_1(E) \to F_n \to \mathrm{Aut}(F_n) \to \mathrm{Out}(F_n) \to 1.$$

The map $F_n \to \mathrm{Aut}(F_n)$ can be seen to be conjugation, and it's kernel, $\pi_1(E)$, is thus trivial. Thus we have proved the following.

Proposition 3.2.1 (see [10], Proposition 1.2)**.** *If $n > 1$ there is a short exact sequence,*

$$1 \to F_n^s \to \Gamma_{n,s} \to \mathrm{Out}(F_n) \to 1$$

The group cohomology $H^i(\Gamma_{n,s})$ admits an action of the symmetric group S_s, and thus defines an FB-module $H^i(\Gamma_{n,\bullet})$ taking the finite set $\mathbf{s}$ to $H^i(\Gamma_{n,s})$. We now show that $H^i(\Gamma_{n,\bullet})$ actually determines an FI-module.

Proposition 3.2.2. *Fix $i, n \geq 0$. $H^i(\Gamma_{n,\bullet})$ is an FI-module.*

Proof. It suffices to describe a functorial way to assign to an injection $\phi \in \mathrm{Hom}_{\mathsf{FI}}(\mathbf{t}, \mathbf{s})$ a linear map $H^i(\Gamma_{n,t}) \to H^i(\Gamma_{n,s})$. Fix a graph $X_{n,s}$ obtained by attaching s hairs to the rose R_n at its single vertex. The marked points are the 1-valent vertices of $X_{n,s}$, which we identify with $\mathbf{s}$. Define $X_{n,t}$ similarly and identify its marked points with $\mathbf{t}$. Pick a homotopy equivalence $f : X_{n,t} \to X_{n,s}$ that acts as ϕ on the marked points of $X_{n,t}$; that is, the marked point x in $X_{n,t}$ should be sent to the marked points $\phi(x)$ in $X_{n,s}$. Let g be a self homotopy equivalence of $X_{n,s}$ fixing the hairs so that g determines an element of $\Gamma_{n,s}$. Now the conjugate fgf^{-1} is a self homotopy equivalence of $X_{n,t}$ fixing its marked points $\mathbf{t}$ and as such determines an element h of $\Gamma_{n,t}$.

This procedure determines a map $\Gamma_{n,s} \to \Gamma_{n,t}$ that depends on the choice of f. However, up to conjugation by $\Gamma_{n,t}$ this element h only depends on ϕ, and thus induces a well-defined map on cohomology, which doesn't see inner automorphisms. □

Remark 3.2.3. If we choose f that it only permutes the hairs of $X_{n,t}$ (i.e., induces the identity on π_1) then the map $\Gamma_{n,s} \to \Gamma_{n,t}$ can be thought of as *forgetting* the $s-t$ points not in the image of ϕ, and relabelling the hairs according to ϕ.

Remark 3.2.4. It is perhaps tempting to draw on the FI structure at the level of groups and try and make $\Gamma_{1,\bullet}$ a (contravarient) functor from FI to the category of groups; a co-FI-group in the language of [7]. Indeed, to an injection $\phi \in \mathrm{Hom}_{\mathsf{FI}}(t,s)$ we described maps from $\Gamma_{n,s}$ to $\Gamma_{n,t}$ in the proof of Proposition 3.2.2. However, the element fgf^{-1} in $\Gamma_{n,t}$ depended on the choice of homotopy equivalence f and as such it is false that $\Gamma_{n,\bullet}$ forms a co-FI-group in general. That being said, it is straightforward to show that $\Gamma_{1,\bullet}$ does form a co-FI-group. We have no need for that result here, and therefore don't do so.

A useful observation is that the structure maps are always injective.

Proposition 3.2.5. *The structure maps of $H^i(\Gamma_{n,\bullet})$ are injective.*

Proof. It suffices to show that the maps $\Gamma_{n,s} \to \Gamma_{n,t}$ used to build the structure maps are split. In the case $t \neq 0$ this is a straightforward adaptation of a result in [10] (see Proposition 1.2) where they prove that there is a splitting $\Gamma_{n,s} \to \Gamma_{n,s-k}$ when $k < s$. The remaining case where $t = 0$ is also dealt with in [10], (see Theorem 1.4) where they prove that natural map $\mathrm{Aut}(F_n) \to \mathrm{Out}(F_n)$ splits on the level of (rational) (co)homology. □

3.2.1 The case in rank 1

In rank 1 the situation is somewhat simpler and we don't need to appeal to a spectral sequence argument to witness Theorem 3.1.2.

Proposition 3.2.6. *As FI-modules*

$$H^i(\Gamma_{1,\bullet}) = \begin{cases} P(1^i) & i \textit{ even} \\ 0 & i \textit{ odd} \end{cases}$$

and as such $H^i(\Gamma_{1,\bullet})$ satisfies Theorem 3.1.2.

Proof. The S_s-module structure of the cohomology in rank 1 was computed in [10], Proposition 2.7 to be

$$H^i(\Gamma_{1,s}) = \begin{cases} P_{(s-i,1^i)} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$$

We need only consider the even case, where we have an FI-module which behaves like $P(1^i)$ when evaluated at any finite set. The fact that the irreducible decomposition contains exactly one irreducible at each finite set implies that the structure maps either agree with those of $P(1^i)$ or are zero. Proposition 3.2.5 says that the structure maps are injective, so we have an equality of FI-modules $H^i(\Gamma_{1,\bullet}) = P(1^i)$ when i is even. As for satisfying Theorem 3.1.2, $P(1^i)$ has stability degree ≤ 1 and weight $\leq i$ by Lemma 2.3.23, as desired. □

3.3 Higher ranks: A spectral sequence argument

With the rank 1 case taken care of we proceed to prove Theorem 3.1.2 in higher rank by establishing a spectral sequence of FI-modules converging to $H^i(\Gamma_{n,\bullet})$. Throughout this section fix $i \geq 0$ and $n \geq 2$.

Lemma 3.3.1. *There is a spectral sequence of* FI*-modules*

$$E_2^{pq} = \bigoplus_{|\lambda|=q,\lambda_1\leq n} C_{p,\lambda} \otimes M(\lambda) \Rightarrow_p H^i(\Gamma_{n,\bullet}),$$

converging to the FI*-module* $H^i(\Gamma_{n,\bullet})$*, where* $C_{p,\lambda}$ *is a constant* FI*-module depending only on* p *and* λ*.*

Proof. Let

$$C_{p,\lambda} = H^p(\mathrm{Out}(F_n); \mathbb{S}_{\lambda'} H)$$

where $H := H^1(F_n) = \Bbbk^n$ and where $\mathbb{S}_{\lambda'}$ is the Schur functor corresponding to the conjugate partition λ'. Define E_2^{pq} as in the statement of the lemma.

We will show that, when evaluated at the finite set $\mathbf{s} = \{1, \ldots, s\}$, this gives the second page of the Leray-Serre spectral sequence of groups associated to the short exact sequence

$$1 \to F_n^s \to \Gamma_{n,s} \to \mathrm{Out}(F_n) \to 1$$

from Proposition 3.2.1. In other words, we will show that

$$(E_2^{pq})_s = H^p(\mathrm{Out}(F_n); H^q(F_n^s)) \Rightarrow_p H^{p+q}(\Gamma_{n,s})$$

as a spectral sequence of groups. Functoriality of the Leray-Serre spectral sequence will complete the proof.

First observe that, by the Künneth formula, $H^q(F_n^s) = H^{\wedge q} \circledast P_{(s-q)}$ as an S_s-module (this is proved carefully in [10], Lemma 2.4). We have,

$$H^p(\mathrm{Out}(F_n); H^q(F_n^s)) = H^p(\mathrm{Out}(F_n); H^{\wedge q} \circledast P_{(s-q)}).$$

The $\mathrm{Out}(F_n)$ action on $H^{\wedge q}$ factors through a $GL_n(\mathbb{Z})$ action. We decompose using Schur-Weyl duality giving,

$$\begin{aligned} H^p(\mathrm{Out}(F_n); H^{\wedge q} \circledast P_{(s-q)}) &= \bigoplus_{|\lambda|=q} H^p(\mathrm{Out}(F_n); \mathbb{S}_\lambda H \otimes P_{\lambda'} \circledast P_{(s-q)}) \\ &= \bigoplus_{|\lambda|=q} H^p(\mathrm{Out}(F_n); \mathbb{S}_\lambda H) \otimes P_{\lambda'} \circledast P_{(s-q)} \\ &= \bigoplus_{|\lambda|=q} H^p(\mathrm{Out}(F_n); \mathbb{S}_\lambda H) \otimes M(\lambda')_s \end{aligned}$$

where λ' is the conjugate partition of λ. Now observe that $\mathbb{S}_\lambda H = 0$ if λ has more than n rows by the character formula (for details see [13]). Therefore λ' has at most n columns, i.e., $\lambda'_1 \leq n$. Therefore

$$H^p(\mathrm{Out}(F_n); H^q(F_n^s)) = \bigoplus_{\substack{|\lambda'|=q \\ \lambda'_1 \leq n}} H^p(\mathrm{Out}(F_n); \mathbb{S}_\lambda H) \otimes M(\lambda')_s = (E_2^{pq})_s.$$

Swapping λ with λ' completes the proof. □

We now describe the stability type and weight of the FI-module E_2^{pq}.

Lemma 3.3.2. *E_2^{pq} has stability type $(0, n)$ and weight q.*

Proof. $C_{p,\lambda}$ are constant FI-modules and thus do not contribute to weight or stability type. $M(\lambda)$ has stability type $(0, \lambda_1)$ and weight $|\lambda|$ by Proposition 2.3.22. E_2^{pq} is obtained by summing over partitions $\lambda \vdash q$ with at most n columns. In particular, each λ satisfies $\lambda_1 \leq n$ and $|\lambda| = q$. □

We are ready to give the spectral sequence argument.

Lemma 3.3.3. *The FI-modules E_k^{pq} on the kth page of the spectral sequence have stability degree n.*

Proof. We denote the stability type of E_k^{pq} by $(\mathcal{I}_k^{pq}, \mathcal{S}_k^{pq})$. We use Lemma 2.3.25 and the fact that the spectral sequence is concentrated in the first quadrant to inductively produce bounds on stability type in subsequent pages.

On the 2nd page all terms E_2^{pq} have stability type at most $(0, n)$. To compute the terms on the third page we use the differentials d_2^{pq} of bidegree $(2, -1)$. We indicate the stability type of terms for convenience.

$$\begin{array}{ccccc} E_2^{p-2,q+1} & \xrightarrow{d_2^{p+2,q-1}} & E_2^{pq} & \xrightarrow{d_2^{pq}} & E_2^{p+2,q-1} \\ (0, \mathcal{S}_2^{p-2,q+1}) & & (0,n) & & (0,n) \end{array}$$

where

$$\mathcal{S}_2^{p-2,q+1} = \begin{cases} 0 & p = 0,1 \\ n & p \geq 2 \end{cases}$$

depending on whether or not $E_2^{p-2,q+1}$ is in the first quadrant. Now Lemma 2.3.25 gives us that

$$\mathcal{I}_3^{pq} = \max(\mathcal{S}_2^{p-2,q+1}, \mathcal{I}_2^{pq}) = \begin{cases} 0 & p = 0,1 \\ n & p \geq 2 \end{cases}$$

$$\mathcal{S}_3^{pq} = \max(\mathcal{S}_2^{pq}, \mathcal{I}_2^{p+2,q-1}) = n.$$

We proceed similarly with the inductive step. We have

$$\begin{array}{ccccc} E_k^{p-k,q+k-1} & \xrightarrow{d_k^{p+k,q-k+1}} & E_k^{pq} & \xrightarrow{d_k^{pq}} & E_k^{p+k,q-k+1} \\ (*, \mathcal{S}_k^{p-k,q+k-1}) & & (\mathcal{I}_k^{pq}, n) & & (n, *) \end{array}$$

and can finally conclude that

$$\mathcal{I}_{k+1}^{pq} = \max(\mathcal{S}_k^{p-k,q+k-1}, \mathcal{I}_k^{pq}) = \begin{cases} 0 & p = 0,1 \\ n & p \geq 2 \end{cases}$$

$$\mathcal{S}_{k+1}^{pq} = \max(\mathcal{S}_k^{pq}, \mathcal{I}_k^{p+k,q-k+1}) = n,$$

since $\mathcal{S}_k^{p-k,q+k-1} = n$ unless $p < k$ when it is 0, and $\mathcal{I}_k^{pq} = n$ unless $p = 0, 1$ when it is 0. □

We are now ready to prove our main result.

Proof of Theorem 3.1.2. First observe that $E_\infty^{p,i-p}$ has weight $\leq i$ and stability degree $\leq n$. Indeed $E_\infty^{p,i-p}$ is a subquotient of $E_2^{p,i-p}$ and thus has weight $\leq i - p \leq i$ by Lemma 3.3.2, and $E_\infty^{p,i-p}$ has stability degree n by Lemma 3.3.3.

We have that E_2^{pq} is a first quadrant spectral sequence of FI-modules converging to the FI-module $H^{p+q}(\Gamma_{n,\bullet})$. This tells us that there exists a natural filtration of $H^i(\Gamma_{n,\bullet})$ whose graded quotients are $E_\infty^{p,i-p}$ and so, by Proposition 2.3.25, $H^i(\Gamma_{n,\bullet})$ has stability degree n. Moreover, since weight is preserved under extensions, $H^i(\Gamma_{n,\bullet})$ has weight $\leq i$. □

Remark 3.3.4. In [27] Jiménez Rolland develops a general framework for dealing with spectral sequences of FI-modules. In particular, the description given in the proof of Lemma 3.3.1 shows that $H^q(F_n^s)$ has weight and stability degree $\leq q$. Thus, in the notation of [27] Theorem 5.3 we have shown that $\beta = 1$ and that $H^i(\Gamma_{n,\bullet})$ has weight $\leq i$ and stability type $(2i, i)$. We note that this recovers the representation stability bounds of [10] upon which we just improved.

CHAPTER 4

DECOMPOSING SCHUR FUNCTORS ON FREE LIE ALGEBRAS

Certain coefficients $c_{\lambda\mu} \in \mathbb{N}$, indexed by pairs of partitions λ, μ, naturally arise in the study of the Johnson homomorphism of the mapping class group. They can be thought of as describing the decomposition of Schur functors on the free Lie algebra $\mathcal{L}(V)$ into Schur functors on V itself (see Section 4.1). In this section we present an algorithm computing the coefficients $c_{\lambda\mu}$. Our approach is to reinterpret the coefficients as counting solutions to a certain combinatorial problem we call *decomposition puzzles*. In particular, we prove the following theorem.

Theorem 4.2.21. *The coefficient $c_{\lambda\mu}$ counts the number of (weighted) solutions to (μ, λ) decomposition puzzles.*

This combinatorial description provides a discretisation of the problem into several steps outlined in Fig. 4.1. By analysing the computational complexity of each step we are able to make key optimisations to the algorithm. In so doing we are able to compute $257,049$ coefficients, extending the known range of coefficients by a factor of over 750.

At a high level, a solution to a (μ, λ) decomposition puzzle can be represented as a path from μ to λ.

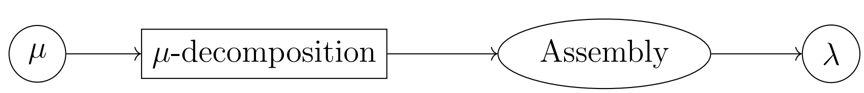


Figure 4.1: Path representing the steps involved in solving a (μ, λ) decomposition puzzle.

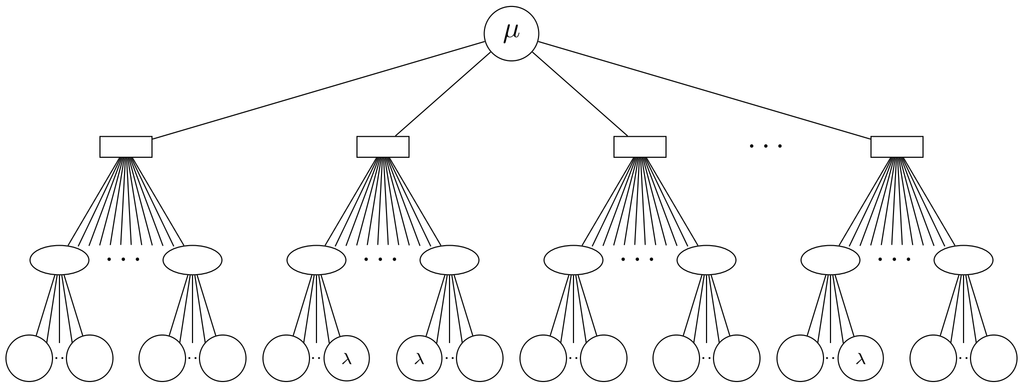


Figure 4.2: Tree representation of all decomposition puzzles. The root here is labelled by the partition μ and Theorem 4.2.21 reinterprets $c_{\lambda\mu}$ as counting the number of leaves labelled by λ. Shape analysis prunes unwanted assemblies (nodes at depth 2), avoiding much unwanted computation.

We collect all such paths into a tree (see Fig. 4.2), whence Theorem 4.2.21 reinterprets $c_{\lambda\mu}$ as counting the number of its leaves labelled by λ. The major hurdle in computing $c_{\lambda\mu}$ is a combinatorial explosion arising in the number of possible assemblies of a given μ-decomposition as the size of μ grows (Eq. 4.7). Our key optimisation is the so called *shape analysis* of a μ-decomposition (Section 4.2.4) which allows us to more efficiently search the leaves of the tree by fixing the degree of the target partition λ in question.

With the algorithm in hand, we turn to the analysis of the data. Visualizing the data appropriately we notice clustering patterns among the coefficient data (see Fig. 4.3). One striking observation is a stability pattern akin to the representation stability of Church-Farb-Ellenberg [9, 7]. In particular, their theory of FI-modules has strong parallels with the stability patterns that emerge from our coefficient data. It is these parallels that lead to a new representation theoretic framework akin to that FI-modules. We develop this in Chapter 5.

The source code for our algorithm is publicly available on GitHub:

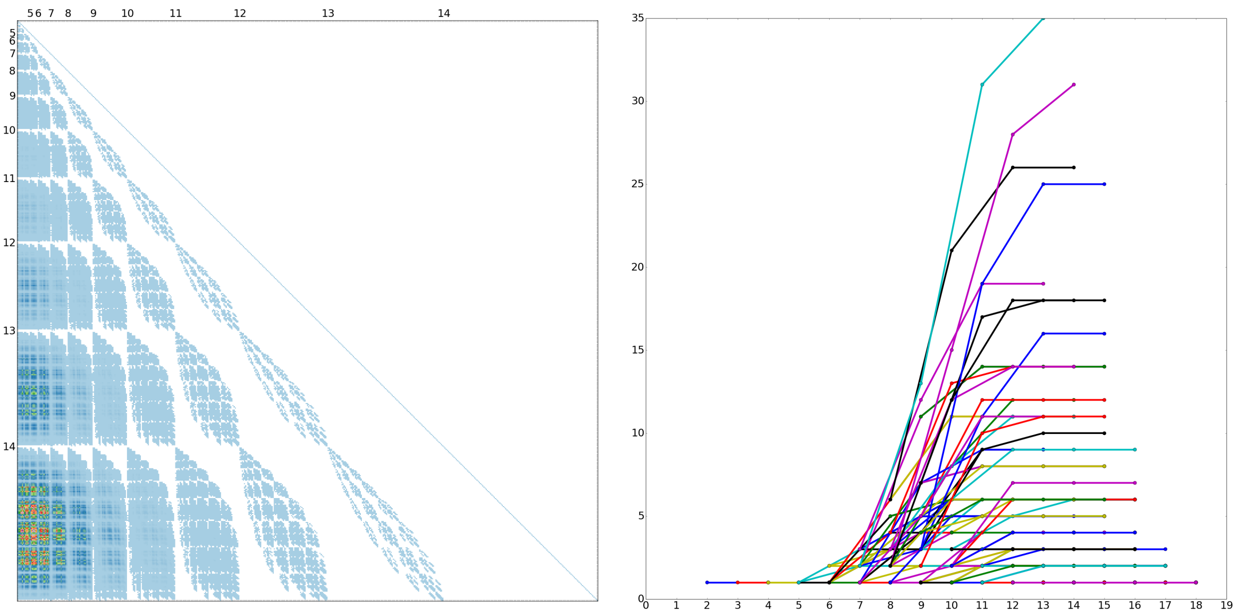


Figure 4.3: Two visualisations of the coefficient data computed by our algorithm. The patterns emerging from these visualisations are suggestive of a stability phenomenon akin to FI-modules. The plot on the left is a representation of all 257,049 coefficients computed by our algorithm. The plot on the right shows the evolution of those coefficients in a certain *stable direction*. See Section 4.4 for more detail.

`https://github.com/aminsaied/composition_factors`

4.1 Coefficients arising in the study of $\mathcal{T}(V)^{\otimes n}$

In [11] they describe an action of $\mathrm{Aut}(F_n)$ on $H^{\otimes n}$ for any cocommutative Hopf algebra H. In particular, the case $H = \mathcal{T}(V)$ is shown to be closely related to the cokernel of the Johnson homomorphism.

An important step in this vein is thus to determine structure of the $\mathcal{T}(V)^{\otimes n}$. In this section we leverage the close relationship between the tensor algebra $\mathcal{T}(V)$ and the free Lie algebra $\mathcal{L}(V)$ to translate this problem into determining certain structure coefficients,

$$c_{\lambda\mu} \in \mathbb{N},$$

arising from applying Schur functors $\mathbb{S}_\mu$ to the free Lie algebra.

A word on notation There are two flavors of tensor product running throughout this section. Those tensor products *internal* to the tensor algebra $\mathcal{T}(V)$, which we denote by $\cdot$, and those that are *external* which we denote with $\otimes$. So, for example, we write $v_1 \cdot v_2, w_1 \cdot w_2 \cdot w_3 \in \mathcal{T}(V)$ and $v_1 \cdot v_2 \otimes w_1 \cdot w_2 \cdot w_3 \in \mathcal{T}(V)^{\otimes 2}$.

The tensor algebra $\mathcal{T}(V)$ admits a natural grading, which extends to a grading on $\mathcal{T}(V)^{\otimes n}$. This degree can be described simply as counting the number of elements of V appearing, so for example,

$$\deg(v_1 \cdot v_2 \otimes w_1 \cdot w_2 \cdot w_3) = 5,$$

where $v_1 \cdot v_2 \in V^{\otimes 2}$ and $w_1 \cdot w_2 \cdot w_3 \in V^{\otimes 3}$.

Definition 4.1.1. Let $\mathcal{T}(V)_d^{\otimes n}$ denote the degree d part of $\mathcal{T}(V)^{\otimes n}$.

Remark 4.1.2. The $\mathrm{Aut}(F_n)$ action on $\mathcal{T}(V)^{\otimes n}$ described in [11] is seen to be *degree preserving*, and thus induces an action of $\mathrm{Aut}(F_n)$ on the degree d part $\mathcal{T}(V)_d^{\otimes n}$. Moreover this action is readily seen to commute with the natural action of $GL(V)$, and thus $\mathcal{T}(V)_d^{\otimes n}$ has the structure of an $(\mathrm{GL}(V), \mathrm{Aut}(F_n))$-bimodule.

Lemma 4.1.3. *Let $d \in \mathbb{N}$. There is an isomorphism of* $\mathrm{GL}(V)$*-modules,*

$$\mathcal{T}(V)_d^{\otimes n} \cong V^{\otimes d} \otimes \mathrm{Sym}^d(\mathbb{k}^n).$$

Proof. There is a natural isomorphism between $\mathrm{Sym}^d(\mathbb{k}^n)$ and the vector space formally spanned by symbols $(a_1|\cdots|a_n)$ where a_i are non-negative integers that sum to d. The isomorphism is given by

$$v_{11}\cdots v_{1a_1} \otimes \cdots \otimes v_{n1}\cdots v_{na_n} \mapsto v_{11}\cdots v_{1a_1}\cdots v_{n1}\cdots v_{na_n} \otimes (a_1|\cdots|a_n)$$

One should think of the a_i's as describing the positions of the external tensor product symbols. □

Fix $n \in \mathbb{N}$ a partition λ and define,

$$M_\lambda^n := P_\lambda \otimes \mathrm{Sym}^d(\mathbb{k}^n).$$

We are able to decompose the degree d part $\mathcal{T}(V)_d^{\otimes n}$ in terms of these M_λ^n as follows.

Lemma 4.1.4. *As a* $(\mathrm{GL}(V), \mathrm{Aut}(F_n))$*-bimodule,* $\mathcal{T}(V)_d^{\otimes n}$ *decomposes as*

$$\mathcal{T}(V)_d^{\otimes n} \cong \bigoplus_{\lambda \vdash d} \mathbb{S}_\lambda(V) \otimes M_\lambda^n.$$

Proof. Schur-Weyl duality says that

$$V^{\otimes d} \cong \bigoplus_{\lambda \vdash d} \mathbb{S}_\lambda(V) \otimes P_\lambda,$$

which in turn gives,

$$\mathcal{T}(V)_d^{\otimes n} \cong V^{\otimes d} \otimes \mathrm{Sym}^d(\mathbb{k}^n) \cong \bigoplus_{\lambda \vdash d} \mathbb{S}_\lambda(V) \otimes P_\lambda \otimes \mathrm{Sym}^d(\mathbb{k}^n) \cong \bigoplus_{\lambda \vdash d} \mathbb{S}_\lambda(V) \otimes M_\lambda^n.$$

□

Theorem 4.1.5. *The* $\mathrm{Aut}(F_n)$*-module* M_λ^n *admits an increasing filtration of length at most* $|\lambda|$*. Moreover, the associated graded* $\mathrm{gr}\, M_\lambda^n$ *is a* $\mathrm{GL}_n(\mathbb{Z})$*-module.*

Proof. The tensor algebra $\mathcal{T}(V)$ is the universal enveloping algebra of the free Lie algebra $\mathcal{T}(V) \cong \mathcal{U}(\mathcal{L}(V))$. As such it admits an increasing filtration, and the PBW theorem gives the isomorphism,

$$\mathcal{T}(V)^i / \mathcal{T}(V)^{i-1} \cong \mathrm{Sym}^i(\mathcal{L}(V)).$$

This filtration induces a filtration on $\mathcal{T}(V)^{\otimes n}$, and it follows that its associated graded

$$\mathrm{gr}(\mathcal{T}(V)^{\otimes n}) = \bigoplus_{i \geq 1} (\mathcal{T}(V)^{\otimes n})^i / (\mathcal{T}(V)^{\otimes n})^{i-1} \cong \mathrm{Sym}^*(\mathcal{L}(V) \otimes \mathbb{k}^n).$$

Furthermore, we have that (for example see [13]),

$$\mathrm{Sym}^i(\mathcal{L}(V) \otimes \Bbbk^n) \cong \bigoplus_{\lambda \vdash i} \mathbb{S}_\lambda(\mathcal{L}(V)) \otimes \mathbb{S}_\lambda(\Bbbk^n),$$

Putting this together we obtain the isomorphism of $\mathrm{GL}(V) \times \mathrm{GL}_n(\mathbb{Z})$-modules,

$$\mathrm{gr}(\mathcal{T}(V)^{\otimes n}) \cong \bigoplus_{\lambda} \mathbb{S}_\lambda(\mathcal{L}(V)) \otimes \mathbb{S}_\lambda(\Bbbk^n).$$

The Schur functor $\mathbb{S}_\lambda(\mathcal{L}(V))$ can be expressed as a sum of Schur functors of V as,

$$\mathbb{S}_\lambda(\mathcal{L}(V)) \cong \bigoplus_{\mu} c_{\lambda\mu} \mathbb{S}_\mu(V).$$

Regrouping terms we have,

$$\mathrm{gr}(\mathcal{T}(V)^{\otimes n}) \cong \bigoplus_{\lambda,\mu} c_{\lambda\mu} \mathbb{S}_\mu(V) \otimes \mathbb{S}_\lambda(\Bbbk^n).$$

The filtration of $\mathcal{T}(V)^{\otimes n}$, together with the decomposition,

$$\mathcal{T}(V)^{\otimes n} \cong \bigoplus_{\mu} \mathbb{S}_\mu(V) \otimes M_\mu^n,$$

induces a filtration of the modules M_μ^n. Comparing the two decompositions we see that,

$$\mathrm{gr}^i(M_\mu^n) \cong \bigoplus_{\lambda \vdash i} c_{\lambda\mu} \mathbb{S}_\lambda(V),$$

which is an isomorphism of $\mathrm{GL}_n(\mathbb{Z})$-modules. □

Putting this together we arrive at the following definition of the structure coefficients $c_{\lambda\mu}$.

Definition 4.1.6. Fix partitions λ, μ. The structure coefficients $c_{\lambda\mu} \in \mathbb{N}$ are defined as the multiplicities,

$$\mathbb{S}_\mu\left(\mathcal{L}(V)\right) \cong \bigoplus_{\lambda} c_{\lambda\mu} \mathbb{S}_\lambda(V), \tag{4.1}$$

In other words, we define $c_{\lambda\mu}$ as the number of times $\mathbb{S}_\lambda(V)$ appears in the decomposition of $\mathbb{S}_\mu(\mathcal{L}(V))$.

4.2 Decomposition puzzles

In this section we recast the definition of the structure coefficients $c_{\lambda\mu}$ in combinatorial terms well suited to an algorithmic approach.

In the introduction we represented a decomposition puzzle as a path in a certain tree. We start by expanding that path into a schematic overview of decomposition puzzles. We will go on to describe the component moves in the remainder of the section.

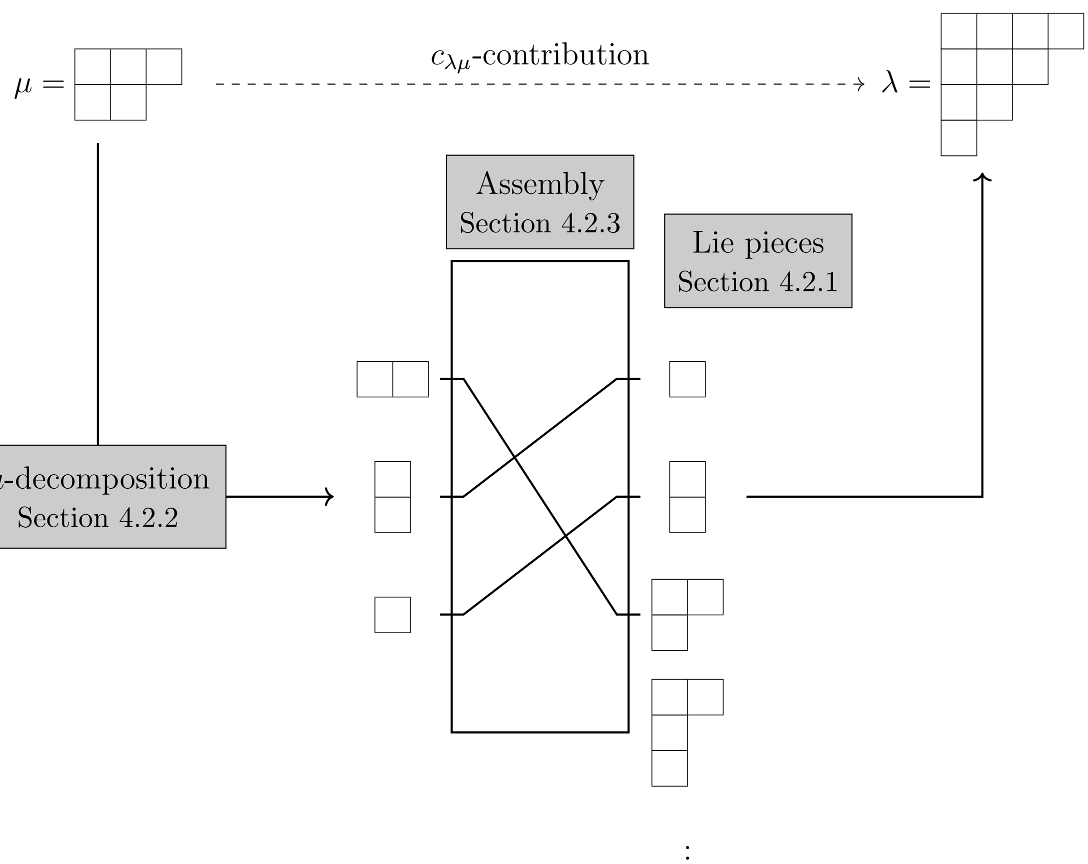


Figure 4.4: A schematic overview of a solution to a decomposition puzzle. This also serves as an example, where in this case we have a partition $\mu = (3, 2)$ decomposing into three partitions $(2), (1, 1)$ and (1). These are then assembled with Lie pieces when finally we arrive at the partition $\lambda = (4, 3, 2, 1)$

4.2.1 Lie pieces

Central to this point of view is the decomposition of the free Lie algebra into its irreducible $\mathrm{GL}(V)$-modules.

Definition 4.2.1. A standard **Young tableaux** of shape $\lambda \vdash n$ is a Young diagram of shape λ filled in (bijectively) with the numbers $\{1, \ldots, n\}$ so that the numbers are increasing along the rows and columns.

Definition 4.2.2. Given a tableaux T of shape λ, define $\mathrm{maj}(T)$ as the sum of i such that $i+1$ lies below i in T.

Example 4.2.3. Let $\lambda = (2, 1, 1) \vdash 4$, then,

$$T = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

is a standard tableaux of shape λ. We have that $\mathrm{maj}(T) = 2 + 3 = 5$.

Theorem 4.2.4 (Stanley)**.** *Let $\lambda \vdash d$. Then the multiplicity of P_λ in Lie_d is given by the number of Young tableaux T of shape λ satisfying maj$(T) \equiv 1 \mod d$.*

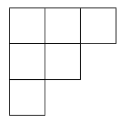
This theorem governs the partitions λ appearing in the irreducible decomposition of the Whitehouse modules Lie_d for all $d > 0$. Moreover, it gives the multiplicity with which each partition appears. We collect all such partitions, counted with multiplicity into an (infinite) collection $\mathbb{L}$ of Lie pieces. Consequently, we can use $\mathbb{L}$ to describe the Whitehouse modules Lie_d and the free Lie algebra $\mathcal{L}_d(V)$.

$$\mathsf{Lie}_d = \bigoplus_{\substack{\lambda \vdash d \\ \lambda \in \mathbb{L}}} P_\lambda \qquad \qquad \mathcal{L}_d(V) = \bigoplus_{\substack{\lambda \vdash d \\ \lambda \in \mathbb{L}}} \mathbb{S}_\lambda(V) \tag{4.2}$$

Definition 4.2.5. A **Lie piece** is a Young diagram appearing in $\mathbb{L}$.

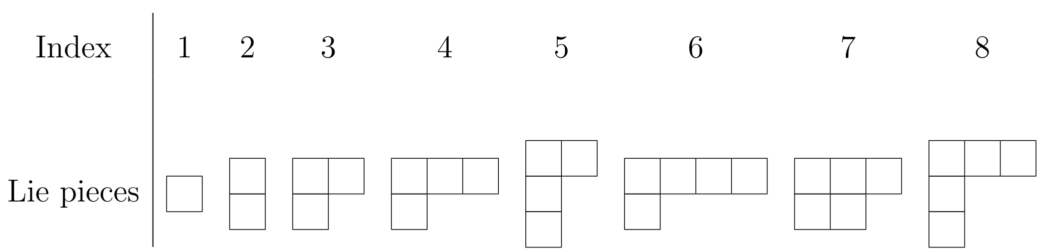
Remark 4.2.6.

1. It is important to note that there are duplicates in the collection of Lie pieces. For example, the term $\mathbb{S}_{[3,2,1]}(V)$ appears with multiplicity 3 in $\mathcal{L}_6(V)$, so there are three copies of



in the collection of Lie pieces.

2. It will be convenient in what follows to fix, once and for all, an order on $\mathbb{L}$. We order the pieces first in increasing size order. If Lie pieces are of the same size then we order the partitions lexicographically (lex order), putting those partitions with largest lex order first. We list the first few terms in $\mathbb{L}$.



The free Lie algebra is an infinite-dimensional vector space, a fact which does not lend itself well to the kinds of finite computation we are interested in here. In practice we therefore work with a truncated, finite-dimensional piece of the free Lie algebra.

Definition 4.2.7. (Truncation.) The truncation (of degree d) of the free Lie algebra is,

$$\mathcal{L}_{\leq d}(V) := \bigoplus_{i \leq d} \mathcal{L}_i(V).$$

The truncation of Lie pieces, denoted $\mathbb{L}_{\leq d}$, is the subcollection of $\mathbb{L}$ consisting of Young diagrams with size at most d.

Remark 4.2.8. The truncation $\mathcal{L}_{\leq d}(V)$ is also known as the *free* d-step nilpotent Lie algebra on V.

Remark 4.2.9. We point out that the number of Lie pieces in $\mathbb{L}_{\leq d}$ grows rapidly as a function of d. Here are the sizes of the first ten truncations.

d	1	2	3	4	5	6	7	8	9	10
$\lvert\mathbb{L}_{\leq d}\rvert$	1	2	3	5	10	22	55	149	439	1388

It is the rapid growth indicated here that causes the dramatic slowdown in computing $c_{\lambda\mu}$ for partitions λ, μ of large degree (see (4.7) for example).

4.2.2 μ-decompositions

Definition 4.2.10. Let μ be a partition. A **μ-decomposition** is a collection of (not necessarily distinct) partitions $(\mu_1, \ldots, \mu_k)$ such that,

$$|\mu| = |\mu_1| + \cdots + |\mu_k|.$$

We consider two μ-decompositions $(\mu_1, \ldots, \mu_k), (\mu'_1, \ldots, \mu'_l)$ equivalent if $k = l$ and there exists some permutation of the indices $\sigma \in \mathrm{Sym}_k$ such that the ordered collections agree:

$$(\mu'_1, \ldots, \mu'_k) = (\mu_{\sigma(1)}, \ldots, \mu_{\sigma(k)}).$$

We tacitly impose this equivalence relation, and choose representatives of equivalence classes as those μ-decompositions $(\mu_1, \ldots, \mu_k)$ where $|\mu_i| \geq |\mu_{i+1}|$, and if $|\mu_i| = |\mu_{i+1}|$ we order them lexicographically.

Iterated Littlewood-Richardson coefficients

Definition 4.2.11. The iterated Littlewood-Richardson coefficient $L^{\mu}_{\mu_1 \cdots \mu_k}$ of a partition μ and a k-tuple of partitions $(\mu_1, \ldots, \mu_k)$ is defined, for $k > 2$, in terms of usual

Littlewood-Richardson coefficients as,

$$L^{\mu}_{\mu_1\cdots\mu_k} := \sum_{\nu_1,\ldots,\nu_{k-2}} L^{\mu}_{\mu_1\nu_1} L^{\nu_1}_{\mu_2\nu_2} \cdots L^{\nu_{k-2}}_{\mu_{k-1}\mu_k}, \tag{4.3}$$

where ν_i are partitions with sizes given below:

1. $|\nu_1| = |\mu| - |\mu_1|$
2. $|\nu_i| = |\nu_{i-1}| - |\mu_i|$ for $2 \leq i \leq k-2$

For convenience we extend the definition to collections of size $k = 1, 2$ by declaring that the coefficient $L^{\mu}_{\mu_1\mu_2}$ is the usual Littlewood-Richardson coefficient, and that the coefficient $L^{\mu}_{\mu_1}$ is the indicator function on the partition μ.

Definition 4.2.12. We say a μ-decomposition $(\mu_1, \ldots, \mu_k)$ is **good** if $L^{\mu}_{\mu_1\cdots\mu_k} > 0$.

There is a recursive algorithm computing these iterated Littlewood-Richardson coefficients, and thus determining if a given μ-partition is good.

4.2.3 Assembly

We now describe assembly; the process by which partitions λ are constructed from a μ-decomposition and a tuple of Lie pieces.

Definition 4.2.13. A **pairing** of a μ-decomposition $D_\mu = (\mu_1, \ldots, \mu_k)$ is a collection of k distinct[1] Lie pieces $L = (l_{i_1}, \ldots, l_{i_k})$ together with a bijection ϕ on the indices of D_μ and of L.

[1]Distinct indices of Lie pieces, as opposed to distinct partitions. The distinction is important as there are multiplicities > 1 appearing in the decomposition of the free Lie algebra.

Algorithm 1: Iterated Littlewood-Richardson Coefficient

Input: A partition μ and an array D of k partitions $[\mu_1, \ldots, \mu_k]$.

Result: Return the iterated Littlewood-Richardson coefficient $L^{\mu}_{\mu_1 \cdots \mu_k}$.

```
ITER_LR(μ, D)
if length D = 1:
    p ← D[0]
    return Indicator 𝕀_μ(p)
elif length D = 2:
    p, q ← D[0], D[1]
    return L^μ_{p,q}
else:
    p ← D[0]
    m ← |μ| − |p|
    c ← 0
    for ν ∈ {ν ⊢ m : ν ⊆ μ}:
        l = L^μ_{pν}
        if l > 0:
            x = ITER_LR(ν, D[1 :])
            c ← c + l ∗ x
    return c
```

For clarity, we consider *straightening* the pairing by relabelling the Lie pieces according to the bijection ϕ so that μ_j is paired with l_{i_j}. We denote such a pairing by

$$(\mu_1, \ldots, \mu_k) \smile (l_{i_1}, \ldots, l_{i_k}).$$

We depict a pairing, together with its straightened counterpart in Fig. 4.5 below.

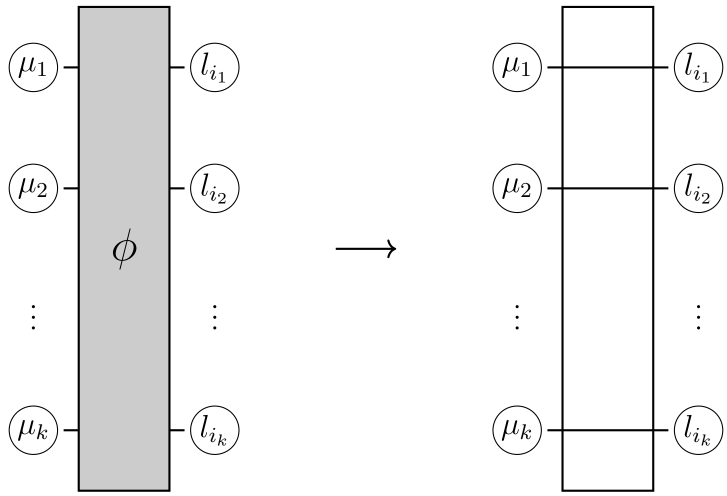


Figure 4.5: On the left we depict a pairing of $(\mu_1, \ldots, \mu_k)$ with a collection of Lie pieces $(l_{i_1}, \ldots, l_{i_k})$. On the right is the straightened version of this pairing, with the indices of the Lie pieces shuffled and relabelled according to the bijection ϕ.

We are now ready to describe the assembly of a (straightened) pairing.

Definition 4.2.14. An **assembly**[2] of a (straightened) pairing,

$$(\mu_1, \ldots, \mu_k) \smile (l_{i_1}, \ldots, l_{i_k}),$$

is the collection of partitions arising in,

$$(\mu_1 \circ l_{i_1}) \otimes (\mu_2 \circ l_{i_2}) \otimes \cdots \otimes (\mu_k \circ l_{i_k}). \tag{4.4}$$

We denote this assembly by,

$$(\mu_1, \ldots, \mu_k) \circledast (l_{i_1}, \ldots, l_{i_k}).$$

Remark 4.2.15. The expression (4.4) is where a lot of the work is being done in computing $c_{\lambda\mu}$. Here we iteratively apply plethysms and tensor products of various partitions. When our partitions are relatively small, this can be done quickly, but as

[2]Here both senses of the word are employed. On the one hand, we think of *assembling* two collections of partitions, and on the other we think of the *assembled* collection of partitions that arise from the construction.

our partitions become large enough it becomes infeasible. There is no getting around this fact, and so our goal is to make the minimal number of applications of $\circledast$ as possible.

The following result forms the basis of our approach to computing $c_{\lambda\mu}$.

Lemma 4.2.16. *Fix a partition $\mu \vdash m$, and a μ-decomposition $(\mu_1, \dots, \mu_k)$. Then any assembly with $(\mu_1, \dots, \mu_k)$ consists of partitions of size at least m. Moreover, if,*

$$(\mu_1, \dots, \mu_k) \smile (l_{i_1}, \dots, l_{i_k})$$

is a pairing, then every partition appearing in its assembly is of size,

$$|\mu_1| \cdot |l_{i_1}| + \dots |\mu_k| \cdot |l_{i_k}|.$$

Proof. The first statement follows immediately from the second. The second is a straightforward consequence of the definition of an assembly as a sequence of plethysms and tensor products. $\square$

In light of this lemma we make the following definition.

Definition 4.2.17. We say an assembly $(\mu_1, \dots, \mu_k) \circledast (l_{i_1}, \dots, l_{i_k})$ has **target-size**,

$$|\mu_1| \cdot |l_{i_1}| + \dots |\mu_k| \cdot |l_{i_k}|.$$

Example 4.2.18. We are ready to give an example of a solution to a decomposition puzzle. Let $\mu = [2, 1]$. Then an example of a good μ-decomposition is,

$$\mu_1 = \square\square \qquad \mu_2 = \square$$

An example of a (straightened) pairing of this μ-decomposition is,

$$l_2 = \begin{array}{c}\square\\ \square\end{array} \qquad l_1 = \square$$

We compute the corresponding assembly of $(\mu_1, \mu_2) \smile (l_2, l_1)$.

$$\mu_1 \circ l_2 = \ydiagram{2} \circ \ydiagram{1,1} = \ydiagram{1,1,1,1} \oplus \ydiagram{2,2} \qquad \mu_2 \circ l_1 = \ydiagram{1} \circ \ydiagram{1} = \ydiagram{1}$$

$$(\mu_1, \mu_2) \circledast (l_2, l_1) = \left(\ydiagram{1,1,1,1} \oplus \ydiagram{2,2} \right) \otimes \ydiagram{1} = \ydiagram{1,1,1,1,1} \oplus \ydiagram{2,1,1,1} \oplus \ydiagram{2,2,1} \oplus \ydiagram{3,2}$$

Observe that all partitions appearing in $(\mu_1, \mu_2) \circledast (l_2, l_1)$ are of size

$$5 = |\mu_1| \times |l_2| + |\mu_2| \times |l_1|.$$

Therefore this assembly has target-size 5. In Fig. 4.6 we show the four paths in the μ-decomposition tree corresponding to the above computation.

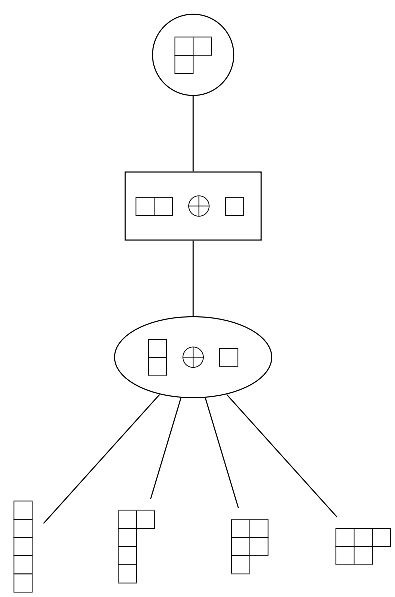


Figure 4.6: An example of four paths in the tree representation of a μ-decomposition puzzle in the case $\mu = (2, 1)$..

We can now formally describe the decomposition puzzle and their solutions.

Definition 4.2.19. A solution $\mathbf{s}$ to a (μ, λ) **decomposition puzzle** is a pairing,

$$(\mu_1, \ldots, \mu_k) \smile (l_{i_1}, \ldots, l_{i_k})$$

of a good μ-decomposition such that λ appears in $(\mu_1, \ldots, \mu_k) \circledast (l_{i_1}, \ldots, l_{i_k})$.

Definition 4.2.20. We say a solution **contributes**,

$$\mathrm{Contrib}(\mathbf{s}) := \alpha \cdot \beta,$$

where α is the iterated Littlewood-Richardson coefficient $L^{\mu}_{\mu_1 \cdots \mu_k}$ and β is the multiplicity with which λ appears in the assembly $(\mu_1, \ldots, \mu_k) \circledast (l_{i_1}, \ldots, l_{i_k})$.

Let $\Sigma = \Sigma_{(\mu,\lambda)}$ denote the set of all distinct solutions to (μ, λ) decomposition puzzles.

Theorem 4.2.21. *The coefficient $c_{\lambda\mu}$ is the weighted sum of all solutions to (μ, λ) decomposition puzzles. That is,*

$$c_{\lambda\mu} = \sum_{\mathbf{s} \in \Sigma_{(\mu,\lambda)}} \mathrm{Contrib}(\mathbf{s}).$$

Proof. From Eq. (4.1), we have that,

$$\mathbb{S}_\mu(\mathcal{L}(V)) \cong \bigoplus_{\lambda} c_{\lambda\mu} \mathbb{S}_\lambda(V).$$

A basic property of Schur functors $\mathbb{S}_\mu$ is that,

$$\mathbb{S}_\mu(A \oplus B) \cong \bigoplus_{\mu_1,\mu_2} L^{\mu}_{\mu_1\mu_2} \mathbb{S}_{\mu_1}(A) \otimes \mathbb{S}_{\mu_2}(B), \tag{4.5}$$

where $|\mu| = |\mu_1| + |\mu_2|$ (see, for example, [13]). It follows from our decomposition of the free Lie algebra into its Lie pieces in (4.2), and by iterative applications of (4.5), that,

$$\mathbb{S}_\mu(\mathcal{L}(V)) \cong \bigoplus L^{\mu}_{\mu_1\cdots\mu_k} \cdot \mathbb{S}_{\mu_1}(\mathbb{S}_{l_{i_1}}(V)) \otimes \cdots \otimes \mathbb{S}_{\mu_k}(\mathbb{S}_{l_{i_k}}(V)), \tag{4.6}$$

where the sum is over all pairings of all μ-decompositions.

By definition the coefficient $c_{\lambda\mu}$ is the multiplicity with which $\mathbb{S}_\lambda(V)$ appears in $\mathbb{S}_\mu(\mathcal{L}(V))$. Consider a summand appearing in the RHS of (4.6) indexed by a pairing,

$$(\mu_1, \ldots, \mu_k) \smile (l_{i_1}, \ldots, l_{i_k}).$$

This pairing is a solution to a (μ, λ) decomposition puzzle if and only if λ appears as a summand in the assembly of the pairing. Moreover, it is easy to see that the multiplicity with which λ appears in this summand is precisely the contribution of that solution. □

We can immediately say something about coefficients $c_{\lambda\mu}$ when $|\lambda| = |\mu|$.

Lemma 4.2.22. *Let λ, μ partitions such that $|\lambda| = |\mu|$. Then,*

$$c_{\lambda\mu} = \begin{cases} 1 & \lambda = \mu \\ 0 & \textit{else} \end{cases}$$

Proof. Let $(\mu_1, \ldots, \mu_k)$ be a μ-decomposition. By Lemma 4.2.16, the size of partitions in an assembly is,

$$|\mu_1| \cdot |l_{i_1}| + \cdots + |\mu_k| \cdot |l_{i_k}|.$$

Furthermore, we have that $|\mu_1| + \cdots + |\mu_k| = |\mu|$. Observe that there is only one Lie piece of size 1, namely,

$$l_1 = \square\,,$$

and so the only way to obtain partitions of size $|\mu|$ in the assembly is if $k = 1$ and $l_{i_1} = l_1$. There is only one μ-decomposition of length 1, μ itself! The result follows. □

With this result in hand we have a potential strategy for computing the coefficients $c_{\lambda\mu}$, namely, enumerate all possible solutions to (μ, λ) decomposition puzzles. The problem, as we outline below, is that the naive approach is computationally infeasible. In the next section we highlight the source of this infeasibility, and provide a workaround that considers the *shape* of a decomposition.

4.2.4 Shape analysis

Before we define the shape of a decomposition, we outline the the problem it seeks to address. Fix partitions μ, λ. By Theorem 4.2.21, our strategy for computing $c_{\lambda\mu}$ is to find all solutions to (μ, λ) decomposition puzzles. Fix a μ-decomposition $(\mu_1, \ldots, \mu_k)$. *A priori*, finding corresponding solutions involves checking the assemblies of all pairings $(l_{i_1}, \ldots, l_{i_k})$ in $\mathbb{L}$. As stated this problem is not even finite! Of course, we don't need to consider all of $\mathbb{L}$. By Lemma 4.2.16 we need only consider Lie parts of size at most $|\lambda| = d$, so we can restrict our search to the truncation $\mathbb{L}_{\leq d}$.

Our problem is now finite, but it is too large. Indeed, we are left to check all possible ordered k-tuples in $\mathbb{L}_{\leq d}$. For each such pairing we form an assembly, which involves computing k plethysms and $(k-1)$ tensor products. All together, the number of computations for the single μ-decomposition $(\mu_1, \ldots, \mu_k)$ is

$$\mathcal{O}\left(\frac{f(d)!}{(f(d)-k)!} \cdot k^2\right), \tag{4.7}$$

where $f : \mathbb{N} \to \mathbb{N}$ is the function taking $d \mapsto |\mathbb{L}_{\leq d}|$.

Remark 4.2.23. There are two major problems with (4.7).

1. (4.7) represents the number of plethysm and tensor products that need to be computed - and these operations (especially the plethysm) are computationally

expensive.

2. The function f grows very quickly (see Remark 4.2.9), causing the expression (4.7) to explode.

We address each of these points in turn in the next two sections.

Avoid unnecessary plethysms and tensor products

The following proposition follows immediately from Lemma 4.2.16 and provides a workaround to Remark 4.2.23 (1).

Proposition 4.2.24. *Fix partitions μ, λ. If $\mathbf{s} = (\mu_1, \ldots, \mu_k) \smile (l_{i_1}, \ldots, l_{i_k})$ is a solution to the (μ, λ) decomposition puzzle, then,*

$$|\mu_1| \cdot |l_{i_1}| + \cdots + |\mu_k| \cdot |l_{i_k}| = |\lambda|. \tag{4.8}$$

Notice that this condition can be checked *without computing plethysms or tensor products*. Our modified strategy therefore is only to check assemblies of pairings for which (4.8) holds.

Definition 4.2.25. The **shape** of μ-decomposition $(\mu_1, \ldots, \mu_k)$ is the partition $\theta \vdash |\mu|$ with parts given by the sizes of its constituent partitions μ_j. That is,

$$\theta = (|\mu_1|, |\mu_2|, \ldots, |\mu_k|)$$

(possibly after reordering). See Fig. 4.7.

The figure below depicts the simplification this analysis affords us.

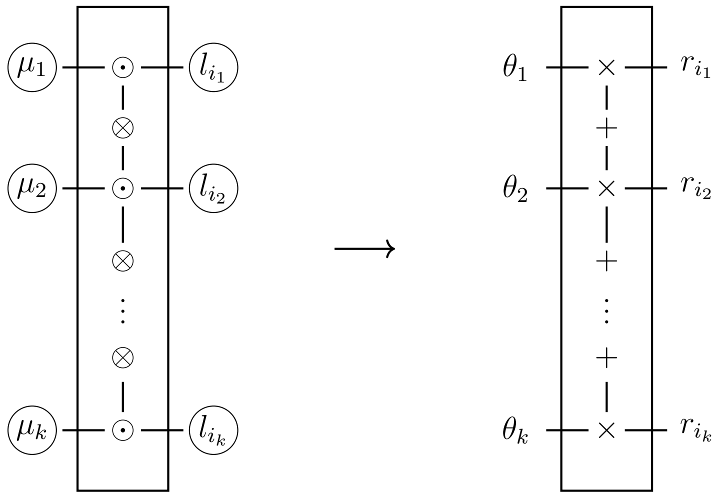


Figure 4.7: On the left we depict a (straightened) pairing of $(\mu_1, \ldots, \mu_k)$ with a collection of Lie pieces $(l_{i_1}, \ldots, l_{i_k})$. On the right is the associated shape partition θ together with the sizes r_i corresponding to the Lie pieces l_i. Notice that plethysms and tensor products on the LHS become multiplications and additions on the RHS (resp.).

Strategy. Our strategy will be to restrict attention to those pairings satisfying (4.8). We describe the algorithm producing such pairings in Algorithm 2. Observe that it is possible for two different μ-decompositions to have the same shape. It is therefore more efficient to find solutions to (4.8) among the set of shapes, and to cache these solutions in a hash table,

$$\{\text{shape} : \text{indices of Lie pieces}\}. \tag{4.9}$$

This strategy means we only compute tensor products and plethysms when their target-size is valid. It therefore addresses Remark 4.2.23 (1), as promised.

Example 4.2.26. To illustrate the scale of savings this makes; when $k = 3$ and $d = 9$, the number of pairings of target size 9 is 148, whereas the number of possible 3-element subsets of $\mathbb{L}_{\leq 9}$ is 84027234. Of course as d increases and as k increases this difference only increases.

Improved upper bound on the size of Lie pieces

We now address the second problem, Remark 4.2.23 (2). Recall that the source of this problem was that the number of Lie pieces of size $\leq d$ grows very quickly as a function of d. Our strategy is to find an improved upper bound on the truncation of Lie pieces.

Definition 4.2.27. Fix a shape $\theta = (\theta_1, \ldots, \theta_k)$ and a degree $d \in \mathbb{N}$. Define $\varphi = \varphi(\theta, d) \in \mathbb{N}$ by,

$$\varphi = \left\lfloor \frac{d - (\theta_1 \cdot r_1 + \cdots + \theta_{k-1} \cdot r_{k-1})}{\theta_k} \right\rfloor$$

where $r_i = |l_i|$ is the size of the i-th Lie piece.

Lemma 4.2.28. *Fix partitions μ, λ and a μ-decomposition of shape θ. Let $\varphi = \varphi(\theta, |\lambda|)$. Then any solution to the (μ, λ) decomposition puzzle involving this μ-decomposition can be found in the truncation,*

$$\mathbb{L}_{\leq \varphi}.$$

Proof. As usual, let r_i denote the size of the i-th Lie piece. Consider the set,

$$X = \{\rho \in \mathbb{L}_{|\cdot|} : \theta_1 \cdot r_1 + \cdots + \theta_{k-1} \cdot r_{k-1} + \theta_k \cdot \rho \leq |\lambda|\}.$$

First observe that by construction $\varphi = \max(X)$. Since the θ_i's are weakly decreasing and the r_i's are weakly increasing, it is clear that,

$$\theta_1 \cdot r_1 + \cdots \theta_k \cdot r_k \tag{4.10}$$

is minimal among $\{\theta_1 \cdot r_{i_1} + \cdots + \theta_k \cdot r_{i_k} : (r_{i_1}, \ldots, r_{i_k}) \in \mathbb{L}^k_{|\cdot|}\}$.

Suppose for a contradiction that there exists a k-tuple $(r_{i_1}, \ldots, r_{i_k}) \in \mathbb{L}^k_{|\cdot|}$ with some $r_{i_j} > \varphi$ such that the assembly-size,

$$\theta_1 \cdot r_{i_1} + \cdots \theta_k \cdot r_{i_k} \leq \lambda.$$

Let $\sigma \in \mathrm{Sym}_k$ be a(ny) permutation of the sizes r_{i_j} such that $r_{\sigma(i_1)} \leq r_{\sigma(i_2)} \leq \cdots \leq r_{\sigma(i_k)}$. Then our contradictory hypothesis is that $r_{\sigma(i_k)} > \varphi$.

We have,

$$\begin{aligned}
|\lambda| &\geq \theta_1 \cdot r_{i_1} + \cdots + \theta_{k-1} \cdot r_{i_{k-1}} + \theta_k \cdot r_{i_k} \\
&\geq \theta_1 \cdot r_{\sigma(i_1)} + \cdots + \theta_{k-1} \cdot r_{\sigma(i_{k-1})} + \theta_k \cdot r_{\sigma(i_k)} \\
&\geq \theta_1 \cdot r_1 + \cdots + \theta_{k-1} \cdot r_{k-1} + \theta_k \cdot r_{\sigma(i_k)}
\end{aligned}$$

where the last inequality follows from the minimality of (4.10). This shows that $r_{\sigma(i_k)} \in X$, contradicting the maximality of φ. □

The upshot of this result is that we can restrict our search for solutions to the smaller set $\mathbb{L}_{\leq\varphi}$. This addresses Remark 4.2.23 (2) as promised.

Example 4.2.29. We demonstrate the scale of improvement afforded by our improved upper bound φ. Consider the shape $\theta = (2, 2, 1)$ and the target-size 9. We see that $\varphi(\theta, 9) = 3$. The number of 3-element subsets of $\mathbb{L}_{\leq 3}$ is 6, whereas the number of 3-element subsets of $\mathbb{L}_{\leq 9}$ is 84027234.

Implementation of shape analysis

We are ready to turn the discussion above into a procedure that we call *shape analysis*.

Fix a target-size $d \in \mathbb{Z}_{>0}$ and a shape $\theta = (\theta_1, \ldots, \theta_k) \vdash m \leq d$. We compute $\varphi = \varphi(\theta, d)$ and then search in $\mathbb{L}_{\leq\varphi}$ for all k-tuples $(l_{i_1}, \ldots, l_{i_k})$ such that,

$$\theta_1 \cdot r_{i_1} + \cdots + \theta_k \cdot r_{i_k} = d,$$

caching the indices $(i_1, \ldots, i_k)$ as we go.

Definition 4.2.30. We refer to such a k-tuple of indices as an **instruction**.

We implement a recursive algorithm computing all instructions for a given shape θ and target-size d. The psuedo-code for this algorithm is given below.

This algorithm caches its results in a hash table. We give that a hash a name.

Definition 4.2.31. (Instructions.) Let $\mathcal{I} = \mathcal{I}(d)$ denote the hash table (of target-size d) mapping shapes θ to the set of instructions computed in Algorithm 2.

Before presenting our algorithm computing composition factors, there is one subtlety that needs to be addressed.

Over counting. Certain cases arise when we can over count the number of solutions to a (μ, λ)-decomposition puzzle. These are best explained by way of an example. Suppose we have a μ-decomposition of shape $[2, 2]$ and we have a target-size of 5. In this case we see that there are two instructions I_1, I_2:

$$I_1 = (1,2) \qquad \rightsquigarrow \qquad \theta_1 \times |\square| + \theta_2 \times \left|\begin{smallmatrix}\square\\\square\end{smallmatrix}\right| = 5$$

$$I_2 = (2,1) \qquad \rightsquigarrow \qquad \theta_1 \times \left|\begin{smallmatrix}\square\\\square\end{smallmatrix}\right| + \theta_2 \times |\square| = 5$$

In the case that the underlying μ-decomposition is,

$$\mu_1 = \begin{smallmatrix}\square\\\square\end{smallmatrix} \quad \mu_2 = \square\square,$$

then both of these instructions give rise to potential solutions. However, suppose the underlying μ-decomposition is as follows.

$$\mu_1 = \begin{smallmatrix}\square\\\square\end{smallmatrix} \quad \mu_2 = \begin{smallmatrix}\square\\\square\end{smallmatrix}$$

In this case, both instructions correspond to the same assembly and any solution arises twice as often as it should.

Algorithm 2: Build instructions

Input: A target-size $d \in \mathbb{Z}_{>0}$ and a shape $\theta \vdash m \leq d$.;

Due to the recursive nature of the algorithm we also pass an instruction I (default empty array []) and a pointer p (default int 0) as input.

Result: We cache completed instructions along the way in a hash table.

```
Compute upper bound φ = φ(θ, d).;
L ← 𝕃_{|≤φ|};
BUILD_INSTRUCTIONS(d, θ, L, I, p)
(base case)
if length θ[p :] = 1:
    for l ∈ L:
        if d = θ[p] · |l|:
            Create new instruction I′ from instruction by adding index of l ∈ 𝕃.
            Cache new instruction I′.
else:
    t ← θ[p]
    p ← p + 1
    for l ∈ L:
        d′ ← d − t · l
        Create new instruction I′ from instruction I by adding index of l ∈ 𝕃.
        BUILD_INSTRUCTIONS(d′, θ, L\{l}, I′, p)
```

It is easy to see that a solution involving the μ-decomposition $(\mu_1, \ldots, \mu_k)$ is over counted in this way if and only if it contains repeated partitions $\mu_i = \mu_j$. Moreover, we can explicitly calculate the size of the over-count.

Definition 4.2.32. Let $(\mu_1, \ldots, \mu_k)$ be a μ-decomposition and let $\{\nu_1, \ldots, \nu_t\}$ be the set of its distinct partitions. Say that ν_i appears in the μ-decomposition n_i times. Define the **over-count factor** of $(\mu_1, \ldots, \mu_k)$ as,

$$\text{over}(\mu_1, \ldots, \mu_k) = (n_1! \cdots n_l!)^{-1}.$$

In the implementation of our algorithm we will account for over counting by computing the over-count factor. Concretely, the contribution of a given solution $\mathbf{s}$ to a (μ, λ)-decomposition puzzle with μ-decomposition $(\mu_1, \ldots, \mu_k)$ is multiplied by the over-count factor $\text{over}(\mu_1, \ldots, \mu_k)$.

4.3 The algorithm

We are now ready to outline the algorithm computing $c_{\lambda\mu}$. Our strategy is to compute all coefficients $c_{\lambda\mu}$ with $|\lambda| = d$ fixed at once. By Lemma 4.2.22 we already know the coefficients $c_{\lambda\mu}$ in the case $|\mu| = d$. Our algorithm will therefore iterate through all partitions μ of size at most $d - 1$. Fix a partition $\mu \vdash m < d$.

μ-decompositions. We first describe how to generate all possible μ-decompositions. Recall from Section 4.2.4 that many μ-decompositions can have the same underlying shape θ. Fix a shape $\theta = (\theta_1, \ldots, \theta_k) \vdash m$ and form the product,

$$M_\theta := \text{Partitions}_\mu(\theta_1) \times \cdots \times \text{Partitions}_\mu(\theta_k),$$

where,

$$\text{Partitions}_\mu(\theta_i) := \{\mu_i \vdash \theta_i : \mu_i \subseteq \mu\}.$$

Notice that a k-tuple in M_θ is precisely a μ-decomposition of shape θ. We are then left to enumerate the set of distinct k-tuples in M_θ, which we denote X_θ. In our implementation we store this in a hash table.

Definition 4.3.1. (μ-decomposisions.) Let $\mathcal{M} = \mathcal{M}(\mu)$ be the hash table (associated to the partition μ) that maps a shape θ to the set of distinct μ-decompositions X_θ.

Assembly. Fix a shape $\theta \vdash m$. Given a μ-decomposition $(\mu_1, \ldots, \mu_k) \in \mathcal{M}[\theta]$ and an instruction $I = (i_1, \ldots, i_k) \in \mathcal{I}[\theta]$ we need to form the assembly,

$$(\mu_1, \ldots, \mu_k) \circledast I := (\mu_1, \ldots, \mu_k) \circledast (l_{i_1}, \ldots, l_{i_k}).$$

This involves applying a sequence of plethysm and tensor product operations[3]. We then collect all Lie pieces λ appearing in $(\mu_1, \ldots, \mu_k) \circledast I$, together with their multiplicities β. Of course, implementing this assembly involves having a representation for the free Lie algebra.

Definition 4.3.2. (Assembly of instructions.) Let $A = A(\mu_1, \ldots, \mu_k; I)$ denote the set of tuples (λ, β) arising in the assembly $(\mu_1, \ldots, \mu_k) \circledast I$.

[3]We implement our algorithm in SAGE, which has optimised implementations of both plethysm and tensor product.

Algorithm 3: Compute composition factors of fixed degree.

Input: A target-size $d \in \mathbb{Z}_{>0}$.

Result: Compute composition factors $c_{\lambda\mu}$ for all partitions λ of size d.

Initialise all coefficients $c_{\lambda\mu} = 0$ for $\lambda \neq \mu$ and $c_{\lambda\lambda} = 1$.

for $m < d$**:**
    for $\theta \vdash m$**:**
        $\mathcal{I}[\theta] \leftarrow$ `BUILD_INSTRUCTIONS`(d, θ)
    for $\mu \vdash m < d$**:**
        $\theta \leftarrow$ shape of μ
        if $\theta \in \mathcal{I}$**:**
            $\mathcal{M} \leftarrow \mathcal{M}(\mu)$ the hash table of μ-decompositions.
            instructions $\leftarrow \mathcal{I}[\theta]$
            for $(\mu_1, \ldots, \mu_k) \in$ decompositions**:**
                $\alpha \leftarrow$ `ITER_LR` $(\mu, [\mu_1, \ldots, \mu_k])$
                if $\alpha > 0$**:**
                    $A \leftarrow A(\mu_1, \ldots, \mu_k; I)$ the assembly with I.
                    for $(\lambda, \beta) \in A$**:**
                        contribution $\leftarrow \alpha \cdot \beta$
                        over $\leftarrow$ over$(\mu_1, \ldots, \mu_k)$
                        $c_{\lambda\mu}$ += contribution $*$ over

4.4 Data analysis

We are now ready to implement our algorithm. The source code for our implementation is publicly available on GitHub[4]. Recall that the coefficient $c_{\lambda\mu}$ can be regarded as the multiplicity of $\mathbb{S}_\lambda$ in

$$\mathbb{S}_\mu(\mathcal{L}(V)). \tag{4.11}$$

As such we are able to use SAGE's symmetric functions libraries to compute the coefficients $c_{\lambda\mu}$ directly from (4.11) (see Section 4.5.1). We use this as a baseline against which we can measure the performance of our algorithm (see Section 4.5).

The baseline algorithm is only able to compute those composition factors $c_{\lambda\mu}$ where λ is of degree at most 5. See Section 4.5 for running time experiments. The optimisations in our algorithm allow us to extend this considerably and compute all composition factors of degree at most 14.

	Degree	Number of coefficients
Baseline	5	324
Our algorithm	14	257,049

Table 4.1: Comparison of our algorithm's range of computation against the baseline algorithm using SAGE's built-in methods.

We are therefore able to extend the range of computation by a factor of over 750. In the next section we begin analysis of the coefficients by visualising the data.

[4] `https://github.com/aminsaied/composition_factors`

4.4.1 Visualisations

In Fig. 4.8 we display the data computed by our baseline algorithm. The axes are labelled by partitions and the colour of the square in position (μ, λ) is determined by the coefficient $c_{\lambda\mu}$ (as per the colour-bar on the right of the plot).

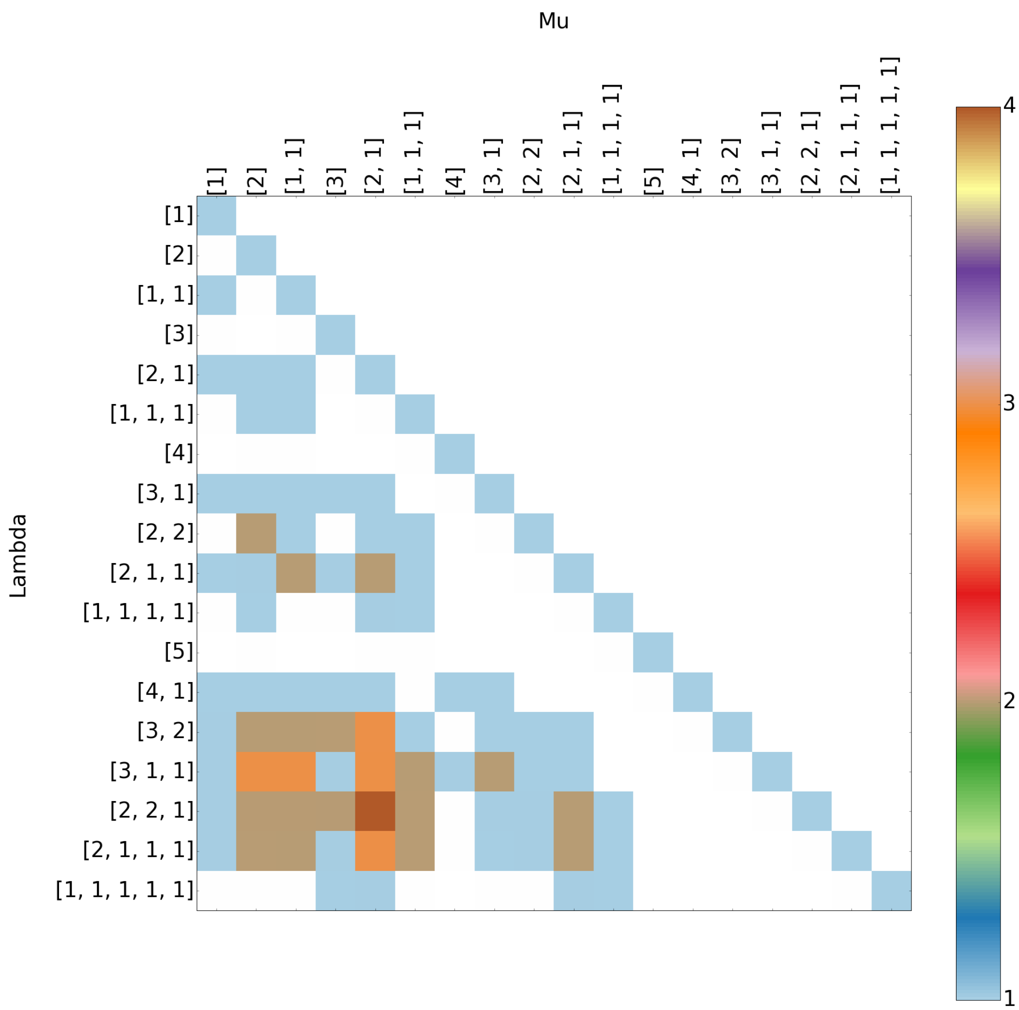


Figure 4.8: Composition factors of degree up to 5. The (μ, λ)-entry is coloured according to the coefficient $c_{\lambda\mu}$, with the scale indicated on the right.

We notice some features even from the small amount of data produced by the baseline algorithm.

1. The diagonal entries are all 1.

2. The matrix is lower-diagonal.
3. If $|\mu| = |\lambda|$ and $\mu \neq \lambda$ then $c_{\lambda\mu} = 0$.

All of these observations are easy to prove and follow immediately from the definition of $c_{\lambda\mu}$. In short, we don't gain much insight from this plot. In Fig. 4.9 we plot $c_{\lambda\mu}$ for all partitions λ, μ of size ≤ 14. As well as being consistent with the previous observations we now notice some more interesting features.

4.4.2 Clustering

At this scale it becomes apparent that there are clusters in the data (see Fig. 4.9). The clusters are confined to rectangular blocks determined by sizes of partitions. Concretely, the pair (μ, λ) lies in the same cluster as (μ', λ') if and only if $|\mu| = |\mu'|$ and $|\lambda| = |\lambda'|$. We therefore refer to the cluster containing (μ, λ) as the $(|\mu|, |\lambda|)$-cluster. A key observation is that the clusters appear to propagate down and to the right. That is, there is a strong similarity between the (m, d)-cluster and the $(m+1, d+1)$-cluster. We investigate this similarity in the next section.

Stabilising plateaus. Fix an initial pair of partitions μ, λ such that $c_{\lambda\mu} > 0$ and consider the process of adding boxes to the top row of these partitions. We introduce some notation.

Definition 4.4.1. For $n \in \mathbb{N}$, let μ^{+n} denote the partition obtained from μ by adding n boxes to the top row of μ.

For example,

$$\mu = \ydiagram{2,2,1} \rightsquigarrow \mu^{+1} = \ydiagram{3,2,1} \rightsquigarrow \mu^{+2} = \ydiagram{4,2,1} \rightsquigarrow \cdots$$

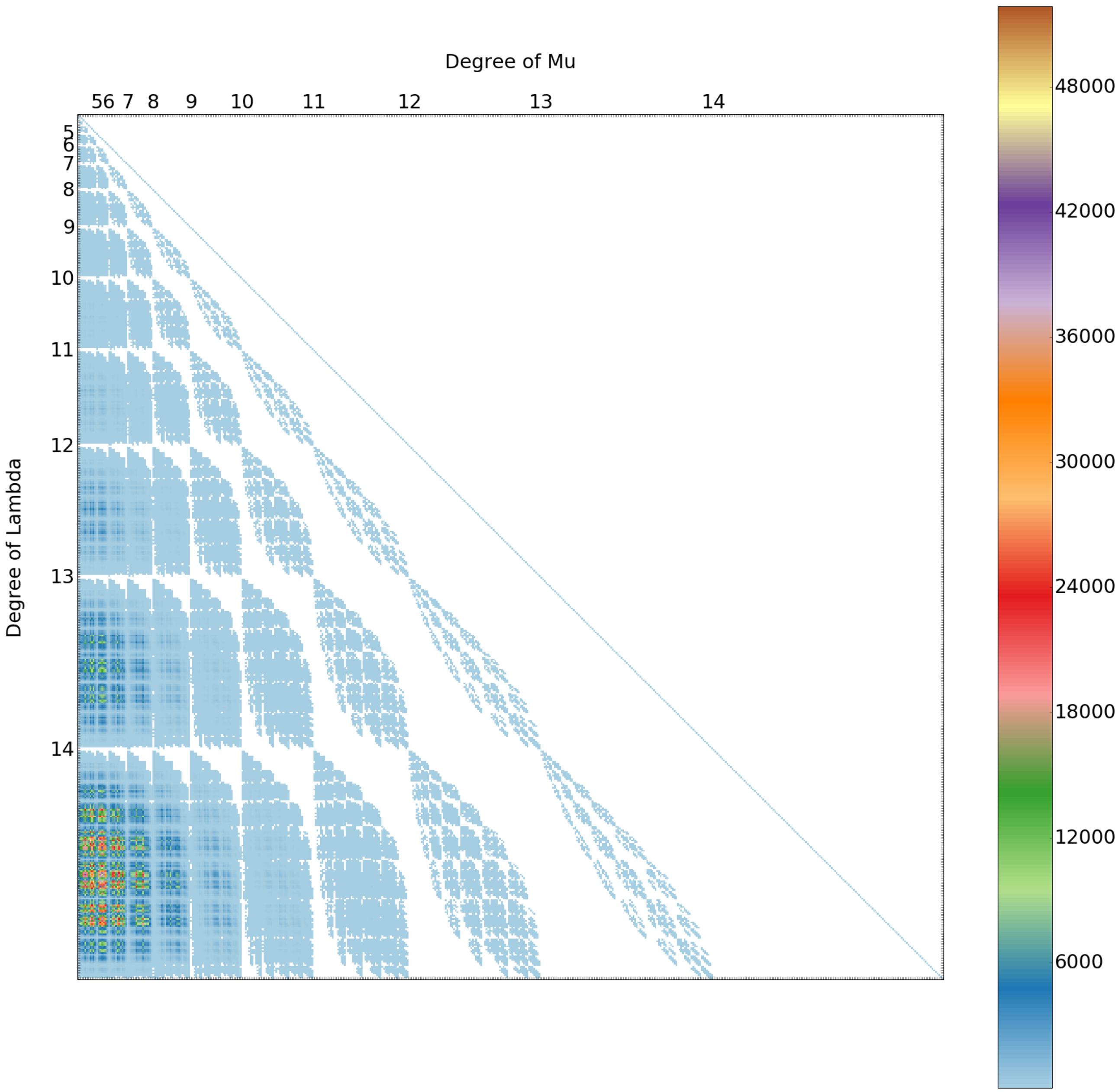


Figure 4.9: A visualization of all coefficients $c_{\lambda\mu}$ of degree < 15. These represents the full range of computations made by our algorithm. For readability we no longer label the partitions on the axes, instead we label the degree (or size) of the partitions at the point at which the degree changes.

Definition 4.4.2. Define the **diagonal push** operation by,

$$\Delta : (\lambda, \mu) \mapsto (\lambda^{+1}, \mu^{+1}).$$

Notice that if a pair of partitions (μ, λ) is in the (m, d)-cluster, then $\Delta(\mu, \lambda)$ lies in the $(m+1, d+1)$-cluster. We are motivated to investigate the behaviour of $c_{\lambda\mu}$ under repeated applications of the operation Δ. In Fig. 4.10 we plot the sequence of coefficients corresponding to,

$$(\lambda, \mu), \Delta(\lambda, \mu), \Delta^2(\lambda, \mu), \ldots$$

for different initial pairs of partitions μ, λ.

The behavior is quite striking. Observe that under the operation of Δ, the coefficients rise to a plateau and stabilize. As the sequences progress the data suggests that the coefficients increase to a point, beyond which the sequences flatten into horizontal tails. In Chapter 5 we formalize this observation by redefining representation stability in this context (Definition 5.5.2). In particular, it will follow from Theorem 5.6.5 that the above stabilization holds in general. In particular, we will prove that, for fixed partitions μ, λ, there exists numbers x, N such that,

$$c_{\lambda^{+r}\mu^{+r}} = x,$$

for all $r \geq N$.

4.5 Running time experiments

In this section we present the results from an experiment comparing the running times of our algorithm against the baseline algorithm. We first describe our experimental set up. All our running time experiments were performed on computer with a 2.3 GHz Intel Core i7 processor and 16GB RAM. Computations were repeated 10 times and

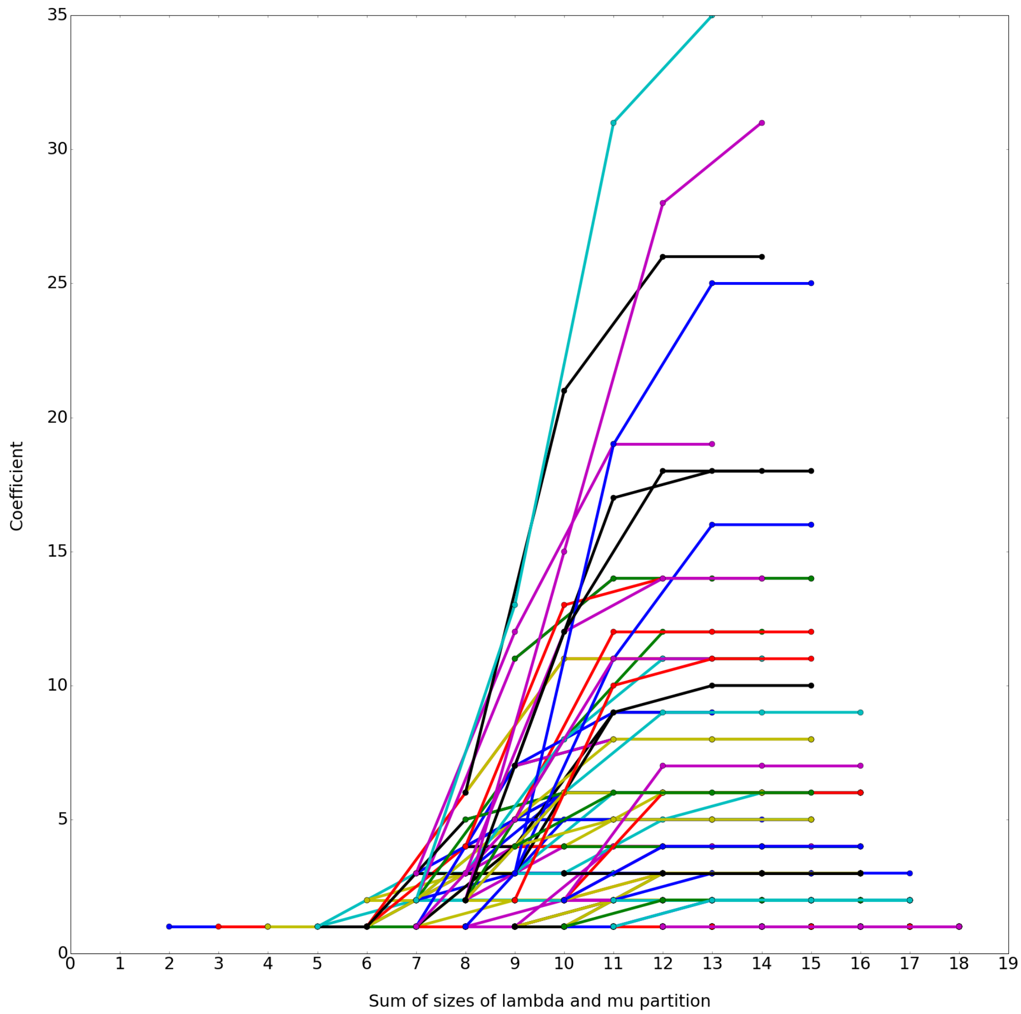


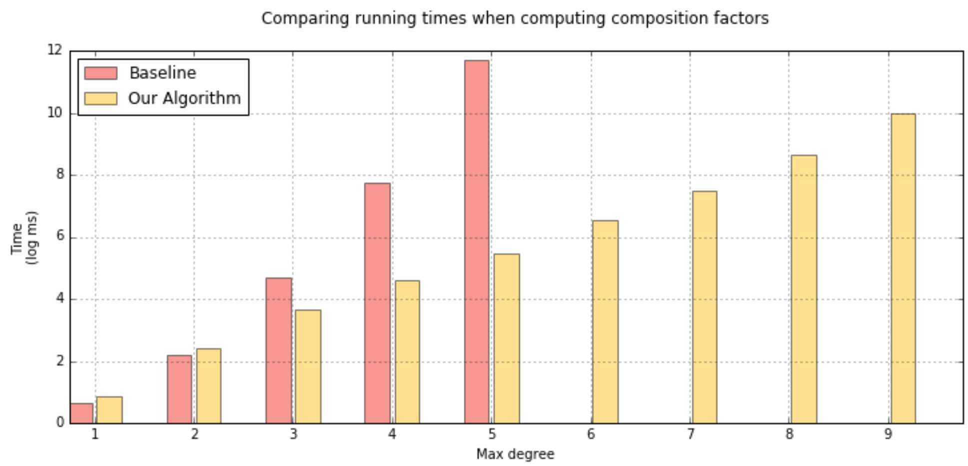
Figure 4.10: The stabilizing effect of adding boxes to the top row. Here we plot data from composition factors of size ≤ 10. Notice the distinctive plateaus.

averaged. We compute all composition factors up to degree d using both the baseline and our own algorithm. Here are the corresponding running times (in seconds).

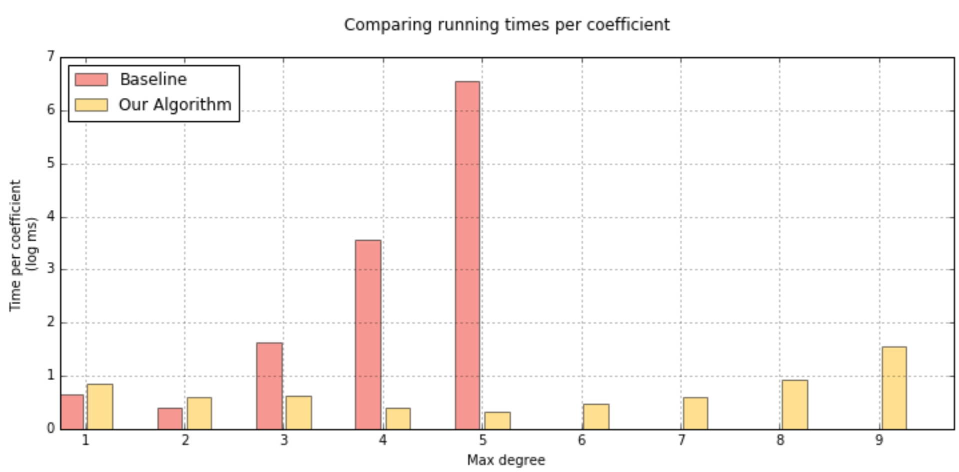
Degree	1	2	3	4	5	6	7	8	9
Baseline	0.00189	0.009	0.107	2.35	119	∞	∞	∞	∞
Our algorithm	0.00237	0.0109	0.0394	0.0979	0.239	0.703	1.82	5.64	21.9

Table 4.2: Running times (in seconds) comparing our algorithm's performance with the baseline.

We plot these against a log-scale to account for the differences in running times in seconds.



Recall that the number of coefficients is increasing rapidly as a function of maximum degree. Below we plot the running times per coefficient. We use the same logarithmic scale in milliseconds.



4.5.1 Baseline Algorithm

We present the baseline algorithm using SAGE's built-in methods. We first assemble the symmetric function `lie` corresponding to the truncation $\mathcal{L}_{\leq n}(V)$. For this we use the same implementation for free Lie algebra class `Lie` (see our source code on GitHub[5]). The key step of this simple algorithm is to compute the plethysm $\mathbb{S}_\mu(\mathcal{L}_{\leq d}(V))$, which is implemented in SAGE as follows.

```
sage: f = s(mu).plethysm(lie)
```

Our baseline algorithm just iterates this over all partitions $\mu \vdash m \leq d$.

Algorithm 4: Baseline algorithm computing $c_{\lambda\mu}$.

Input: A target-size $d \in \mathbb{Z}_{>0}$.

Result: Compute composition factors $c_{\lambda\mu}$ for all partitions λ of size d.

Initialise all coefficients $c_{\lambda\mu} = 0$.
Lie $\leftarrow$ array of Lie pieces of size at most d
for $m \leq d$**:**
    for $\mu \vdash m$**:**
        $f \leftarrow \mu \odot$ Lie
        for $\lambda \in f$**:**
            $c_{\lambda\mu} \leftarrow c_{\lambda\mu} + 1$

[5] https://github.com/aminsaied/composition_factors

CHAPTER 5

THE THEORY OF PD-MODULES

In this section, we develop the representation theory of the category PD (Definition 5.1.1). Just as FI-modules determine sequences

$$\{V_n : n \in \mathbb{N}\}, \tag{5.1}$$

of S_n-modules, so too PD-modules determine collections

$$\{W_{i,n} : i, n \in \mathbb{N}\}, \tag{5.2}$$

of $(\mathrm{S}_i, \mathrm{S}_n)$-bimodules. We start by establishing a framework in parallel with that of the theory of FI-modules. In particular, we will define free PD-modules (Section 5.2.1) and finitely generated PD-modules (Section 5.3). The notion of a finitely generated FI-module is central to that theory, and imposes tight constraints on the sequence (5.1) in the form of representation stability (see Section 2.3.1). To make an analogous statement in the PD setting, we require an analog of representation stability in the context (5.2). We present this in Section 5.5. In Theorem 5.5.3 we show that finitely generated PD-modules give rise to representation stable collections (5.2).

Of course, this notion of representation stability differs from its one-dimensional counterpart. We discuss some of those differences in Section 5.3.1 where we see that the representation theory of finitely generated PD-modules is not as constrained as in the finitely generated FI case.

The following definition will be useful.

Definition 5.0.1. Given an injection $\alpha : X \hookrightarrow X'$ let $\alpha^C \subseteq X'$ denote the compliment of the image of α in X'.

5.1 The category PD

Definition 5.1.1. (The category PD.) Let PD be the category with:

- Objects: Pairs of finite sets (X, Y).
- Morphisms. Given objects $(X, Y), (X', Y') \in \mathrm{ob}(\mathsf{PD})$, a morphism,

  $$\Delta : (X, Y) \to (X', Y'),$$

  is given by the triple,

  $$\Delta = (\alpha, \beta, \gamma),$$

  consisting of,

  (a) An injection $\alpha : X \hookrightarrow X'$,

  (b) An injection $\beta : Y \hookrightarrow Y'$,

  (c) A bijection on the complements of α and β,

  $$\gamma : \alpha^C \to \beta^C.$$

  Composition of morphisms in PD is defined as follows. Given objects (X, Y), (X', Y'), $(X'', Y'') \in \mathrm{ob}(\mathsf{PD})$, and morphisms,

  $$\Delta_1 = (\alpha_1, \beta_1, \gamma_1) \in \mathrm{Hom}_{\mathsf{PD}}((X, Y), (X', Y')),$$

  $$\Delta_2 = (\alpha_2, \beta_2, \gamma_2) \in \mathrm{Hom}_{\mathsf{PD}}((X', Y'), (X'', Y'')),$$

  then,

  $$\Delta = \Delta_2 \circ \Delta_1 \in \mathrm{Hom}_{\mathsf{PD}}((X, Y), (X'', Y'')),$$

  is given by the triple,

  $$\Delta = (\alpha, \beta, \gamma),$$

where $\alpha = \alpha_2 \circ \alpha_1$ and $\beta = \beta_2 \circ \beta_1$ are the respective compositions. To describe the bijection γ, first note that $\alpha^C = \alpha_2(\alpha_1^C) \sqcup \alpha_2^C$ and $\beta^C = \beta_2(\beta_1^C) \sqcup \beta_2^C$. Then we have,

$$\gamma = (\beta_2 \circ \gamma_1 \circ \alpha_2^{-1}) \sqcup \gamma_2 : \alpha_2(\alpha_1^C) \sqcup \alpha_2^C \to \beta_2(\beta_1^C) \sqcup \beta_2^C,$$

which is readily seen to be a bijection.

This map is perhaps most easily understood with the aid of a picture (Fig. 5.1).

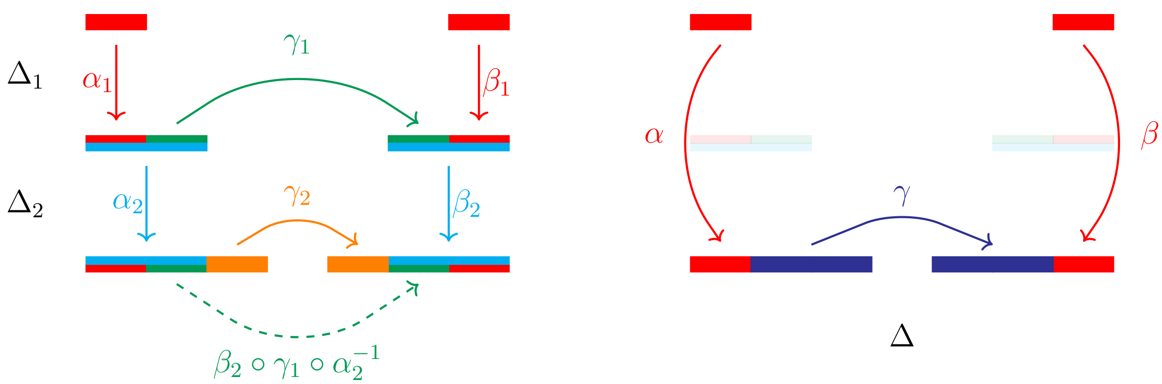


Figure 5.1: Symbolic representation of a composition in the category PD. On the left we have two morphisms Δ_1, Δ_2 of PD, and on the right we see their composition $\Delta = \Delta_2 \circ \Delta_1$.

Remark 5.1.2.

1. By convention there is a unique bijection $\varnothing \to \varnothing$.

2. The endomorphisms $\mathrm{End}_{\mathsf{PD}}(X, Y)$ are isomorphic to $\mathrm{Sym}(X) \times \mathrm{Sym}(Y)$.

3. The category PD is locally small.

4. It will often be convenient to restrict attention to the skeleton category with objects given by pairs $(\mathbf{i}, \mathbf{n})$. To save ink we will often just denote pairs of finite sets by (i, n) where there is no ambiguity.

Remark 5.1.3. Any morphism $f \in \mathrm{Hom}_{\mathsf{PD}}((X,Y),(X',Y'))$ satisifes $|X'|-|X| = |Y'|-|Y|$. If we picture the pair of finite sets (X,Y) as occupying the coordinates $(|X|,|Y|)$ on the plane, then we see that PD-morphims lie on lines of slope 1. This point of view is presented in Fig. 5.2.

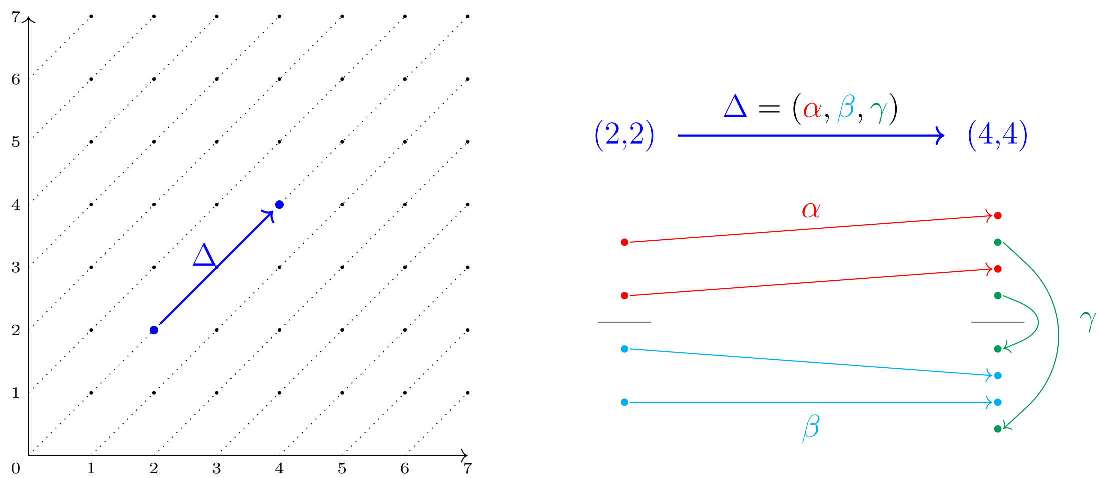


Figure 5.2: A sketch of a morphism in the (skeleton) category PD. On the left is a schematic view of the category PD in which integer lattice points represent (isomorphism classes of) objects in PD. We present an example morphism $\Delta : (2,2) \to (4,4)$ in PD. On the right we see this morphism expanded into its three constituent parts, maps $\alpha : \mathbf{2} \to \mathbf{4}$, $\beta : \mathbf{2} \to \mathbf{4}$ and a map γ defining a bijection on the complements.

We present alternative descriptions of the morphisms in PD. These follow immediately from the definition above.

Lemma 5.1.4.

1. $\mathrm{Hom}_{\mathsf{PD}}((\varnothing,\varnothing),(X,Y)) \cong \mathrm{Hom}_{\mathsf{FB}}(X,Y) \cong \mathrm{Hom}_{\mathsf{FB}}(Y,X)$,

2. $\mathrm{Hom}_{\mathsf{PD}}((S,\varnothing),(X,Y)) \cong \mathrm{Hom}_{\mathsf{FB}}(Y \sqcup S, X)$,

3. $\mathrm{Hom}_{\mathsf{PD}}((\varnothing,T),(X,Y)) \cong \mathrm{Hom}_{\mathsf{FB}}(Y, X \sqcup T)$,

4. $\mathrm{Hom}_{\mathsf{PD}}((S,T),(X,Y)) \cong \{\psi \in \mathrm{Hom}_{\mathsf{FB}}(Y \sqcup S, X \sqcup T) : T \subseteq \psi(Y), \psi(S) \subseteq X\}$.

Proof. The first three statements follow from the last, which follows directly from Definition 5.1.1. $\square$

5.2 Representation theory of PD

The category PD is small, and thus we have an abelian category $\mathsf{PD}\text{-}\mathsf{Mod}$ of its representations. We have pointwise notions for submodule, direct sum, (co)kernel, injection, surjection and quotient as described in Section 2.2.2.

Let $W \in \mathsf{PD}\text{-}\mathsf{Mod}$ and $(X,Y) \in \mathrm{ob}(\mathsf{PD})$. We denote the image of (X,Y) under W simply by $W_{X,Y} \in \mathsf{Vect}$. Since $\mathrm{End}_{\mathsf{PD}}((X,Y)) \cong \mathrm{Sym}(X) \times \mathrm{Sym}(Y)$ we have that each vector space $W_{X,Y}$ is a module over $\mathrm{Sym}(X) \times \mathrm{Sym}(Y)$. In this way, the PD-module W can be thought of as a collection of $(\mathrm{Sym}(X), \mathrm{Sym}(Y))$-bimodules $W_{X,Y}$ together with *diagonal* linear maps compatible with this bimodule structure.

Remark 5.2.1. To emphasis that a PD-module is a functor from a category whose objects are *pairs* of finite sets, we occasionally write $V(\bullet\bullet) : \mathsf{PD} \to \mathsf{Vect}$. Similarly, for a PD-bimodule $V : \mathsf{PD}^{\mathsf{op}} \times \mathsf{PD} \to \mathsf{Vect}$, we emphasis its functoriality by writing,

$$V(\bullet\bullet, \bullet\bullet) : \mathsf{PD}^{\mathsf{op}} \times \mathsf{PD} \to \mathsf{Vect}.$$

Following Definition 2.2.9, the representable functor $\mathsf{R}_{\mathsf{PD}}(\bullet\bullet, \bullet\bullet) \in (\mathsf{PD}^{\mathsf{op}}, \mathsf{PD})\text{-}\mathsf{BiMod}$ sends the pairs of finite sets $(S,T), (X,Y)$ to the free $\Bbbk$-vector space,

$$\Bbbk[\mathrm{Hom}_{\mathsf{PD}}((S,T),(X,Y))].$$

The morphism $f \in \mathrm{Hom}_{\mathsf{PD}^{\mathsf{op}}}((S,T),(S',T')) = \mathrm{Hom}_{\mathsf{PD}}((S',T'),(S,T))$ acts by precomposition, and the morphism $g \in \mathrm{Hom}_{\mathsf{PD}}((X,Y),(X',Y'))$ acts by postcomposition,

$$(f,g)_*(\Delta) : (S',T') \xrightarrow{f} (S,T) \xrightarrow{\Delta} (X,Y) \xrightarrow{g} (X',Y'),$$

for $\Delta \in \mathrm{Hom}_{\mathsf{PD}}((S,T),(X,Y))$. Fixing $(S,T) \in \mathrm{ob}(\mathsf{PD}^{\mathsf{op}})$ results in PD-module.

Definition 5.2.2. Fix an object $(S,T) \in \mathrm{ob}\,\mathsf{PD}^{\mathsf{op}} = \mathrm{ob}(\mathsf{PD})$. Let $N(S,T)_{\bullet\bullet} \in \mathsf{PD}\text{-}\mathsf{Mod}$ denote the evaluation $\mathsf{R}_{\mathsf{PD}}((S,T), \bullet\bullet)$ which sends the object $(X,Y) \in \mathrm{ob}(\mathsf{PD})$ to the free vector space,

$$\Bbbk[\mathrm{Hom}_{\mathsf{PD}}((S,T),(X,Y))],$$

and where PD-morphisms act by post-composition.

Example. We have the following characterizations of these PD-modules.

1. $N(\varnothing,\varnothing)_{X,Y} = \Bbbk[\mathrm{Hom}_{\mathsf{FB}}(X,Y)]$
2. $N(S,\varnothing)_{X,Y} = \Bbbk[\mathrm{Hom}_{\mathsf{FB}}(Y \sqcup S, X)]$
3. $N(\varnothing,T)_{X,Y} = \Bbbk[\mathrm{Hom}_{\mathsf{FB}}(Y, X \sqcup T)]$

5.2.1 Free PD-modules

In Section 2.3.3 we applied the free object paradigm (Section 2.2.6) in the setting of the inclusion,

$$\mathsf{FB} \hookrightarrow \mathsf{FI}$$

to obtain the free FI-modules $\mathrm{Ind}_{\mathsf{FB}}^{\mathsf{FI}}(\bullet)$. We now apply that same paradigm in the setting of the inclusion,

$$\mathsf{FB} \times \mathsf{FB} \hookrightarrow \mathsf{PD}.$$

giving an adjunction,

$$\mathrm{Ind}_{\mathsf{FB}\times\mathsf{FB}}^{\mathsf{PD}}(\bullet) : (\mathsf{FB},\mathsf{FB})\text{-}\mathsf{BiMod} \leftrightharpoons \mathsf{PD}\text{-}\mathsf{Mod} : \mathrm{Res}_{\mathsf{FB}\times\mathsf{FB}}^{\mathsf{PD}}(\bullet).$$

Concretely, we define the inclusion functor,

$$\mathsf{FB} \times \mathsf{FB} \to \mathsf{PD}.$$

The objects of PD and $\mathsf{FB} \times \mathsf{FB}$ coincide, and so the inclusion is the identity on objects. Morphisms in $\mathsf{FB} \times \mathsf{FB}$ are of the form (f,g) for $f \in \mathrm{Hom}_{\mathsf{FB}}(A,A')$ and

$g \in \mathrm{Hom}_{\mathsf{FB}}(B, B')$, for finite sets A, A', B, B'. This is sent to the PD-morphism,

$$(f, g, \varnothing \to \varnothing) \in \mathrm{Hom}_{\mathsf{PD}}((A, B), (A', B')).$$

The restriction functor $\mathrm{Res}^{\mathsf{PD}}_{\mathsf{FB}\times\mathsf{FB}}(\bullet)$ is obtained by precomposition with the inclusion $\mathsf{FB} \times \mathsf{FB} \hookrightarrow \mathsf{PD}$. The induction functor arises as the tensor product,

$$\mathrm{Ind}^{\mathsf{PD}}_{\mathsf{FB}\times\mathsf{FB}}(\bullet) = \bullet \otimes_{\mathsf{FB}\times\mathsf{FB}} \mathsf{R}_{\mathsf{PD}},$$

where R_{PD} is considered a $(\mathsf{FB}^{\mathsf{op}} \times \mathsf{FB}^{\mathsf{op}}, \mathsf{PD})$-bimodule.

Given an $(\mathsf{FB}, \mathsf{FB})$-bimodule W, our goal is to describe the $(\mathrm{S}_i, \mathrm{S}_n)$-module structure on $\mathrm{Ind}^{\mathsf{PD}}_{\mathsf{FB}\times\mathsf{FB}}(W)_{i,n}$. We detailed a similar analysis for free FI-modules in Section 2.3.3. We restrict our attention to the skeletal subcategories of PD and $\mathsf{FB} \times \mathsf{FB}$ with objects of the form $(i, n) \in \mathbb{N} \times \mathbb{N}$.

Observe that an $(\mathsf{FB}, \mathsf{FB})$-bimodule W determines, and is determined by, a collection $\{W_{i,n} : i, n \in \mathbb{N}\}$ of $\mathrm{S}_i \times \mathrm{S}_n$-modules $W_{i,n}$. In this setting we have the following characterization of induced modules. Recall that we occasionally denote $\mathrm{S}_a \times \mathrm{S}_b$ by $S(a, b)$ for legibility.

Lemma 5.2.3. *Let $W \in (\mathsf{FB}, \mathsf{FB})$-$\mathsf{BiMod}$. Then the PD-module $\mathrm{Ind}^{\mathsf{PD}}_{\mathsf{FB}\times\mathsf{FB}}(W)$ satisfies,*

$$\mathrm{Ind}^{\mathsf{PD}}_{\mathsf{FB}\times\mathsf{FB}}(W)_{i,n} \cong \bigoplus_{m\in\mathbb{N}} \mathrm{Ind}^{S(i,n)}_{S(i-m,n-m)\times S(m,m)} W_{i-m,n-m} \boxtimes \Bbbk[\mathrm{S}_m].$$

Proof. As in the proof of Proposition 2.3.15, we assume, without loss of generality, that W is supported in bidegree (l, k) so that $W = W_{l,k}$. We have, by definition of the tensor product, that,

$$\mathrm{Ind}^{\mathsf{PD}}_{\mathsf{FB}\times\mathsf{FB}}(W)_{i,n} \cong W_{l,k} \otimes_{\mathrm{S}_l\times\mathrm{S}_k} \Bbbk[\mathrm{Hom}_{\mathsf{PD}}((l, k), (i, n))].$$

The set $\mathrm{Hom}_{\mathsf{PD}}((l,k),(i,n))$ is empty unless there exists $m \in \mathbb{N}$ such that $i = l+m$ and $n = k+m$. Assume such an m exists. Observe that $\mathrm{Hom}_{\mathsf{PD}}((l,k),(i,n))$ splits as a sum,

$$\mathrm{Hom}_{\mathsf{PD}}((l,k),(i,n)) = \bigoplus_{\substack{I\in\mathcal{I}\\ J\in\mathcal{J}}} \mathrm{Hom}_{\mathsf{PD}}((l,k),(i,n);(I,J)),$$

where $\mathrm{Hom}_{\mathsf{PD}}((l,k),(i,n);(I,J))$ is the subset of $\mathrm{Hom}_{\mathsf{PD}}((l,k),(i,n))$ consisting of those PD-morphisms,

$$(\alpha : \mathbf{l} \hookrightarrow \mathbf{i}, \beta : \mathbf{k} \hookrightarrow \mathbf{n}, \gamma : \alpha^C \leftrightarrow \beta^C),$$

such that $\mathrm{im}(\alpha) = I$ and $\mathrm{im}(\beta) = J$.

Under the natural identification of $\mathrm{Hom}_{\mathsf{PD}}((l,k),(i,n);(I,J))$ with $\mathrm{S}_l \times \mathrm{S}_k \times \mathrm{S}_m$ we can identify,

$$\Bbbk[\mathrm{Hom}_{\mathsf{PD}}((l,k),(i,n);(I,J)],$$

with,

$$\Bbbk[\mathrm{S}_l \times \mathrm{S}_k] \otimes \Bbbk[\mathrm{S}_m].$$

We are thus able to write,

$$\mathrm{Ind}_{\mathsf{FB}\times\mathsf{FB}}^{\mathsf{PD}}(W)_{i,n} \cong \bigoplus_{\substack{I\in\mathcal{I}\\ J\in\mathcal{J}}} W_{l,k} \otimes_{\mathrm{S}_l\times\mathrm{S}_k} \Bbbk[\mathrm{S}_l \times \mathrm{S}_k] \otimes \Bbbk[\mathrm{S}_m] \cong \bigoplus_{\substack{I\in\mathcal{I}\\ J\in\mathcal{J}}} W_{l,k} \otimes \Bbbk[\mathrm{S}_m].$$

Fix $I = \{1,\ldots,l\}$ and $J = \{1,\ldots,k\}$ and consider the corresponding summand,

$$U := \Bbbk[\mathrm{S}_l \times \mathrm{S}_k] \otimes \Bbbk[\mathrm{S}_m].$$

Under the identification with $\Bbbk[\mathrm{Hom}_{\mathsf{PD}}((l,k),(i,n);(I,J))]$ we have that a basis of U consists of elements $(\alpha,\beta,\gamma) \in \mathrm{Hom}_{\mathsf{PD}}((l,k),(i,n))$, so that,

$$\alpha : \mathbf{l} \hookrightarrow \mathbf{i} \qquad \beta : \mathbf{k} \hookrightarrow \mathbf{n} \qquad \gamma : \alpha^C \leftrightarrow \beta^C,$$

and such that $\mathrm{im}(\alpha) = I$ and $\mathrm{im}(\beta) = J$. We see that S_l acts be precomposition with α, S_k acts by precomposition with β, S_i acts by postcomposition with α and S_n acts by postcomposition with β. It follows directly from this description of the action that the stabilizer of U in $\mathrm{S}_i \times \mathrm{S}_n$ is isomorphic to,

$$(\mathrm{S}_l \times \mathrm{S}_k) \times (\mathrm{S}_m \times \mathrm{S}_m).$$

We see that we are in the setting of Lemma 2.1.15, and the result follows. □

Lemma 5.2.4. *Fix finite sets S, T. The* PD*-module $N(S,T)$ is the free* PD*-module,*

$$\mathrm{Ind}_{\mathsf{FB}\times\mathsf{FB}}^{\mathsf{PD}}(\Bbbk[\mathrm{Sym}(S) \times \mathrm{Sym}(T)]).$$

Proof. We have,

$$\mathrm{Ind}_{\mathsf{FB}\times\mathsf{FB}}^{\mathsf{PD}}(\Bbbk[\mathrm{Sym}(S) \times \mathrm{Sym}(T)]) = \Bbbk[\mathrm{Sym}(S) \times \mathrm{Sym}(T)] \otimes_{\mathsf{FB}\times\mathsf{FB}} \mathsf{R}_{\mathsf{PD}},$$

where $\Bbbk[\mathrm{Sym}(S)\times\mathrm{Sym}(T)]$ is considered an FB-module supported in bidegree $(|S|, |T|)$. The tensor product over $\mathsf{FB} \times \mathsf{FB}$ with an FB-module supported in a single bidegree reduces to the tensor product over the group ring $\Bbbk[\mathrm{Sym}(S) \times \mathrm{Sym}(T)]$. We therefore have,

$$\begin{aligned}
&\mathrm{Ind}_{\mathsf{FB}\times\mathsf{FB}}^{\mathsf{PD}}(\Bbbk[\mathrm{Sym}(S)\times \mathrm{Sym}(T)]) \\
&\qquad\qquad \cong \Bbbk[\mathrm{Sym}(S) \times \mathrm{Sym}(T)] \otimes_{\mathrm{Sym}(S)\times\mathrm{Sym}(T)} \Bbbk[\mathrm{Hom}_{\mathsf{PD}}((S,T), \bullet\bullet)] \\
&\qquad\qquad \cong \Bbbk[\mathrm{Hom}_{\mathsf{PD}}((S,T), \bullet\bullet)].
\end{aligned}$$

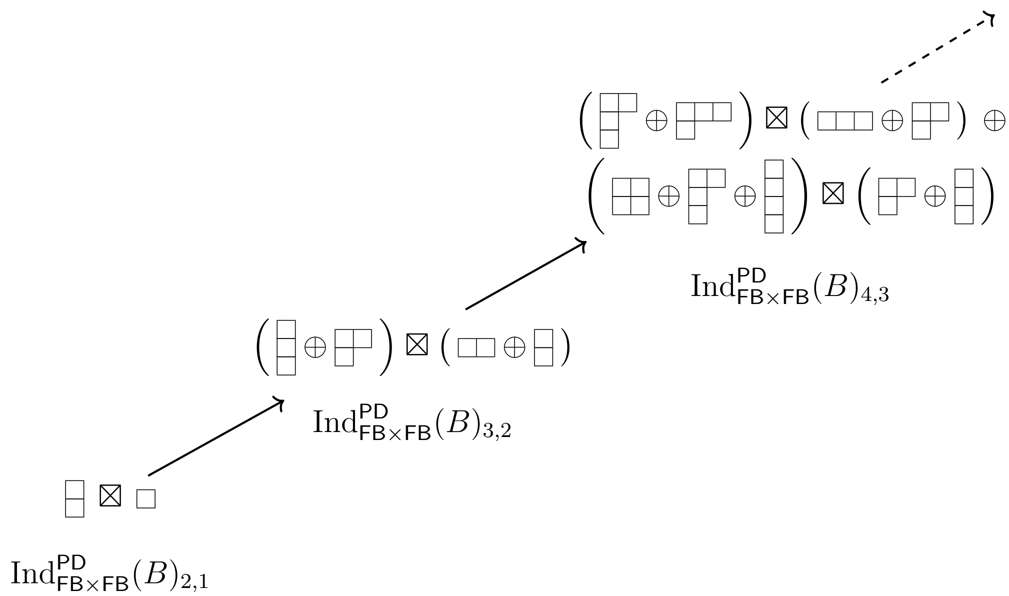
This is precisely the PD-module $N(S,T)$ as required. □

Visualizing PD-modules in terms of Young diagrams. It can be helpful to depict PD-modules in terms of its underlying irreducible representations. Concretely, any PD-module V determines a family $\{V_{i,n} : i, n \in \mathbb{N}\}$ of $S(i,n)$-bimodules. The decomposition of $V_{i,n}$ into its irreducible $S(i,n)$-bimodules can be depicted as an array of young diagrams, as in the example below in which we apply Lemma 5.2.3 using the techniques outlined in Section 2.1.4 to compute the irreducible decompositions.

Example 5.2.5. Let B be the $\mathsf{FB} \times \mathsf{FB}$-bimodule supported in bidegree $(2,1)$ by the irreducible $S(2,1)$-module,

$$\ydiagram{1,1} \boxtimes \ydiagram{1}\,.$$

Then $\mathrm{Ind}_{\mathsf{FB}\times\mathsf{FB}}^{\mathsf{PD}}(B)$ is a PD-module with the following irreducible decomposition (for small bidegrees).



Remark 5.2.6. The free PD-module construction is to be compared with the free FI-module construction where we have,

$$\mathrm{Ind}_{\mathsf{FB}}^{\mathsf{FI}}(W) \cong \mathrm{Ind}_{\mathrm{S}_a \times \mathrm{S}_{n-a}}^{\mathrm{S}_n} W_a \boxtimes \Bbbk,$$

given an FB-module W. Consider the FB-module supported in degree 2 by the irreducible S_2-module,

$$W_2 = \ydiagram{1,1}\,.$$

We have

$$\begin{array}{ccccc} \ydiagram{1,1} & \longrightarrow & \ydiagram{1,1,1} \oplus \ydiagram{2,1} & \longrightarrow & \ydiagram{2,1,1} \oplus \ydiagram{3,1} \\ \mathrm{Ind}_{\mathsf{FB}}^{\mathsf{FI}}(W)_2 & & \mathrm{Ind}_{\mathsf{FB}}^{\mathsf{FI}}(W)_3 & & \mathrm{Ind}_{\mathsf{FB}}^{\mathsf{FI}}(W)_4 \end{array}$$

Notice that these Young diagrams appear on the LHS of the free PD-module of Example 5.2.5 corresponding to the (LHS of the) trivial part $P_{(m)} \boxtimes P_{(m)}$ in the decomposition,

$$\Bbbk[\mathrm{S}_m] \cong \bigoplus_{\mu \vdash m} \left(P_\mu \boxtimes P_\mu \right).$$

5.3 Finite Generation of PD-modules

Here we recast the definitions of finite generation for FI-modules from [7] in the setting of the category PD. Many of the notions introduced by Church-Ellenberg-Farb (in particular see [7], Section 2.3) translate with only slight modification to this setting. The reader familiar with those notions will find no surprises here. Indeed, many of the proofs are so similar as to not bear repeating, and in that event we simply refer to the relevant result in [7].

Definition 5.3.1. (Span) Given $W \in \mathsf{PD}\text{-}\mathsf{Mod}$ let $\Sigma \subseteq \coprod_{i,n} W_{i,n}$ and define $\mathrm{Span}_W(\Sigma)$ to be the minimal sub-PD-module of W containing Σ.

Lemma 5.3.2. *Let $W \in \mathsf{PD}\text{-}\mathsf{Mod}$. Then an element $w \in W_{l,k}$ determines a map $N(l,k) \to W$, the image of which coincides with* $\mathrm{Span}_W(w)$.

Proof. See [7], Lemma 2.3.2. □

Definition 5.3.3. (Finite generation) A PD-module W is *finitely generated* if there exists a surjection,

$$\bigoplus_j N(l_j, k_j) \twoheadrightarrow W.$$

where the sum is finite.

It turns out that in this two dimensional setting, having a notion of finitely generated *along a diagonal* will be useful.

Definition 5.3.4. (Restriction to rank-r.) Let $W \in \mathsf{PD\text{-}Mod}$. Define the restriction of W to rank-r, $W^{(r)} \in \mathsf{PD\text{-}Mod}$ by,

$$W^{(r)}_{i,n} = \begin{cases} W_{i,n} & i = n - r \\ 0 & \text{else} \end{cases}$$

Say that a PD-module is supported in rank-r if $W_{i,n} = W^{(r)}_{i,n}$ for all pairs (i, n).

Definition 5.3.5. (Finitely generated in rank.) A PD-module W is *finitely generated in rank* if the restriction to rank-r $W^{(r)}$ is finitely generated for all $r \in \mathbb{N}$.

5.3.1 Representation instability

One important consequence of finite generation in the category of FI-modules is the highly constrained representation theory of the underlying S_n-modules. In particular, given $V \in \mathsf{FI\text{-}Mod}$ a finitely generated FI-module, there exist a *finite* collection of partitions $\lambda_1, \dots, \lambda_r$ that describe the irreducible decomposition of V_n for any n sufficiently large. Concretely,

$$V_n = \bigoplus_{i=1}^{r} P(\lambda_i)_n,$$

for n sufficiently large. We will contrast this behaviour with an example of a finitely generated PD-module W that needs infinitely many partitions to describe its decomposition into irreducible $\mathrm{S}_i \times \mathrm{S}_n$-modules. First we define the number of corners of a partition.

Definition 5.3.6. (Corners of a partition.) A corner of a partition $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$ is an index $i \in \{1, \dots, k\}$ such that the following two conditions hold (where defined):

1. $\lambda_i > \lambda_{i+1}$

2. $\lambda_i \leq \lambda_{i-1}$

Notice that the number of corners of λ, $\mathrm{Corners}(\lambda)$, counts the number of ways that λ can be obtained from a partition $\mu \vdash n-1$ by adding a single box in accordance with Pieri's rule. For example, the partition $\lambda = (7,4,4,3,1) \vdash 19$ has 4 corners.

$$\begin{ytableau} \; & \; & \; & \; & \; & \; & * \\ \; & \; & \; & \; \\ \; & \; & \; & * \\ \; & \; & * \\ * \end{ytableau}$$

Lemma 5.3.7. *Let $W = N(1,1) \in \mathsf{PD}\text{-}\mathsf{Mod}$. Then*

$$W_{n,n} = \bigoplus_{\lambda,\lambda' \vdash n} m_{\lambda\lambda'} P_\lambda \otimes P_{\lambda'}$$

where the multiplicity $m_{\lambda\lambda'}$ is given by,

$$m_{\lambda\lambda'} = \begin{cases} 0 & |\lambda \cap \lambda'| \leq n-1 \\ 1 & |\lambda \cap \lambda'| = n-1 \\ \mathrm{Corners}(\lambda) & \lambda = \lambda' \end{cases} .$$

This result shows that, unlike for finitely generated FI-modules, this finitely generated PD-module's representation theory cannot be described by finitely many partitions.

Proof. By Lemma 5.2.4 we have,

$$N(1,1) = \mathrm{Ind}_{\mathsf{FB}\times\mathsf{FB}}^{\mathsf{PD}}(\Bbbk[\mathrm{S}_1 \times \mathrm{S}_1]).$$

We have the decomposition into irreducible $\mathrm{S}_1 \times \mathrm{S}_1$-bimodules,

$$\Bbbk[\mathrm{S}_1 \times \mathrm{S}_1] \cong P_{(1)} \boxtimes P_{(1)}.$$

By definition,

$$\begin{aligned}
W_{n,n} &= \mathrm{Ind}_{(\mathrm{S}_1\times\mathrm{S}_1)\times(\mathrm{S}_{n-1}\times\mathrm{S}_{n-1})}^{\mathrm{S}_n\times\mathrm{S}_n}\left(P_{(1)}\boxtimes P_{(1)}\right)\boxtimes \Bbbk[\mathrm{S}_{n-1}] \\
&\cong \bigoplus_{\mu\vdash n-1}\mathrm{Ind}_{(\mathrm{S}_1\times\mathrm{S}_1)\times(\mathrm{S}_{n-1}\times\mathrm{S}_{n-1})}^{\mathrm{S}_n\times\mathrm{S}_n}\left(P_{(1)}\boxtimes P_{(1)}\right)\boxtimes\left(P_\mu\otimes P_\mu\right) \\
&\cong \bigoplus_{\mu\vdash n-1}\mathrm{Ind}_{\mathrm{S}_1\times\mathrm{S}_{n-1}}^{\mathrm{S}_n}\left(P_{(1)}\boxtimes P_\mu\right)\boxtimes\mathrm{Ind}_{\mathrm{S}_1\times\mathrm{S}_{n-1}}^{\mathrm{S}_n}\left(P_{(1)}\boxtimes P_\mu\right)
\end{aligned}$$

Notice that,

$$\mathrm{Ind}_{\mathrm{S}_1\times\mathrm{S}_{n-1}}^{\mathrm{S}_n}\left(P_{(1)}\otimes P_\mu\right)\cong\bigoplus P_\lambda$$

where the sum is over all $\lambda\vdash n$ such that λ is obtained from μ by adding one box in accordance with Pieri's rule. The result follows. $\square$

Example 5.3.8. We compute $W_{3,3}$ directly. Notice that

$$\mathrm{Ind}_{\mathrm{S}_1\times\mathrm{S}_2}^{\mathrm{S}_3}\left(\ydiagram{1}\otimes\ydiagram{2}\right)\cong\ydiagram{3}\oplus\ydiagram{2,1}$$

and

$$\mathrm{Ind}_{\mathrm{S}_1\times\mathrm{S}_2}^{\mathrm{S}_3}\left(\ydiagram{1}\otimes\ydiagram{1,1}\right)\cong\ydiagram{2,1}\oplus\ydiagram{1,1,1}$$

Therefore,

$$\begin{aligned}
W_{3,3} &\cong \left[\left(\ydiagram{3}\oplus\ydiagram{2,1}\right)\otimes\left(\ydiagram{3}\oplus\ydiagram{2,1}\right)\right] \\
&\quad\oplus\left[\left(\ydiagram{2,1}\oplus\ydiagram{1,1,1}\right)\otimes\left(\ydiagram{2,1}\oplus\ydiagram{1,1,1}\right)\right]. \\
&\cong\left(\ydiagram{3}\otimes\ydiagram{3}\right)\oplus\left(\ydiagram{2,1}\otimes\ydiagram{2,1}\right)^{\oplus 2}\oplus\left(\ydiagram{1,1,1}\otimes\ydiagram{1,1,1}\right) \\
&\quad\oplus\left(\ydiagram{3}\otimes\ydiagram{2,1}\right)\oplus\left(\ydiagram{2,1}\otimes\ydiagram{3}\right)\oplus\left(\ydiagram{1,1,1}\otimes\ydiagram{2,1}\right)\oplus\left(\ydiagram{2,1}\otimes\ydiagram{1,1,1}\right)
\end{aligned}$$

which is seen to agree with the multiplicities $m_{\lambda\lambda'}$.

5.4 Endofunctors on FI-Mod arising from PD-modules

Although we cannot hope for such strong representation theoretic constraints on finitely generated PD-modules, as demonstrated by this example, we can still say something. Our strategy will be to construct from a finitely generated PD-module, a finitely generated FI-module.

Let $W \in \mathsf{PD}\text{-}\mathsf{Mod}$. Fix a finite set Y. There is an $\mathsf{FB}^{\mathsf{op}}$-module,

$$W(\bullet, Y) : \mathsf{FB}^{\mathsf{op}} \to \mathsf{Vect},$$

sending the finite set X to the vector space $W(X,Y)$, and sending the morphism $f \in \mathrm{Hom}_{\mathsf{FB}^{\mathsf{op}}}(X, X') = \mathrm{Hom}_{\mathsf{FB}}(X', X)$ to the map,

$$\Delta_f^{(1)} := (f : X' \hookrightarrow X, 1_Y : Y \to Y, \varnothing \to \varnothing). \tag{5.3}$$

Given an FI-module V, we can define the tensor product,

$$V \otimes_{\mathsf{FB}} W(\bullet, Y) \in \mathsf{Vect},$$

where V is considered an FB-module by restriction. To the finite set Y we associate the vector space,

$$(V \mathbin{\widehat{\otimes}} W)(Y) := V \otimes_{\mathsf{FB}} W(\bullet, Y).$$

This determines an FB-module.

Lemma 5.4.1. *The assignment $Y \mapsto (V \mathbin{\widehat{\otimes}} W)(Y)$ determines an FB-module.*

Proof. The morphism $g \in \mathrm{Hom}_{\mathsf{FB}}(Y, Y')$ determines a linear map,

$$g_* : (V \mathbin{\widehat{\otimes}} W)(Y) \to (V \mathbin{\widehat{\otimes}} W)(Y'),$$

as follows. The vector space $(V \mathbin{\widehat{\otimes}} W)(Y)$ is the quotient of,

$$\bigoplus_{X \in \mathrm{ob}(\mathsf{FB})} V(X) \otimes W(X, Y),$$

in which,

$$v_X \otimes f^*(w_{X'Y}) \quad \text{is identified with} \quad f_*(v_X) \otimes w_{X'Y},$$

for all $v_X \in V_X, w_{X'Y} \in W(X', Y), f \in \mathrm{Hom}_{\mathsf{FB}}(X, X')$ for all $X, X' \in \mathrm{ob}(\mathsf{FB})$. The map g determines a PD-morphism $\Delta_g \in \mathrm{Hom}_{\mathsf{PD}}((X,Y),(X,Y'))$ where,

$$\Delta_g^{(2)} = (1_X : X \to X, g : Y \to Y', \varnothing \to \varnothing),$$

where 1_X is the identity on X. This defines a map,

$$\bigoplus_{X \in \mathrm{ob}(\mathsf{FB})} V(X) \otimes W(X,Y) \to \bigoplus_{X \in \mathrm{ob}(\mathsf{FB})} V(X) \otimes W(X,Y'),$$

sending $v \otimes w \to v \otimes \Delta_g^{(2)}(w)$. It remains to verify that this map factors through the quotient,

$$\bigoplus_{X \in \mathrm{ob}(\mathsf{FB})} V(X) \otimes W(X,Y) \twoheadrightarrow (V \mathbin{\widehat{\otimes}} W)(Y).$$

This follows from the relation,

$$\Delta_f^{(1)} \circ \Delta_g^{(2)} = \Delta_g^{(2)} \circ \Delta_f^{(1)},$$

where $\Delta_f^{(1)}$ is as in Eq. 5.3. □

Lemma 5.4.2. *Let $V \in$ FI-Mod and $W \in$ PD-Mod. The FB-module $V \mathbin{\widehat{\otimes}} W$ can be promoted to an FI-module.*

Proof. Let $f \in \mathrm{Hom}_{\mathsf{FI}}(Y, Y')$. We define the map,

$$f_* : (V \mathbin{\widehat{\otimes}} W)(Y) \to (V \mathbin{\widehat{\otimes}} W)(Y')$$

as follows. Let $\psi = \psi_X : X \hookrightarrow X \sqcup f^C$ be the canonical inclusion. Notice that ψ induces a map

$$\psi_* : V(X) \to V(X \sqcup f^C).$$

In addition, $\psi^C = f^C$ and there is a bijection 1_{f^C} on the compliments of ψ and f. That is,

$$\Delta = \Delta_X = (\psi, f, 1_{f^C}) \in \mathrm{Hom}_{\mathsf{PD}}((X,Y),(X \sqcup f^C, Y')).$$

Putting this together, for any $X \in \mathrm{ob}\,\mathsf{FB}$, we have a canonical map, induced by $f \in \mathrm{Hom}_{\mathsf{FI}}(Y,Y')$,

$$\psi_* \otimes \Delta_* : V(X) \otimes W(X,Y) \to V(X \sqcup f^C) \otimes W(X \sqcup f^C, Y').$$

For convenience, we call this map $F_{f,X}$. This induces a map on the quotient $(V \otimes W)(Y) \to (V \otimes W)(Y')$ and it remains to show this map is well-defined. In particular, it suffices to show that the following diagram commutes,

$$\begin{tikzcd}
V(X) \otimes W(X,Y) \arrow[r, "(\psi_X)_* \otimes (\Delta_X)_*"] \arrow[d, "\phi_* \otimes \phi_*"] & V(X \sqcup f^C) \otimes W(X \sqcup f^C, Y') \arrow[d, "\xi_* \otimes \xi_*"] \\
V(X') \otimes W(X',Y) \arrow[r, "(\psi_{X'})_* \otimes (\Delta_{X'})_*"] & V(X' \sqcup f^C) \otimes W(X' \sqcup f^C, Y')
\end{tikzcd}$$

where $\xi = \phi \sqcup 1_{f^C} \in \mathrm{Hom}_{\mathsf{FB}}(X \sqcup f^C, X' \sqcup f^C)$. The commutativity of this diagram follows from the commutativity of,

$$\begin{tikzcd}
X \arrow[r, "\psi_X"] \arrow[d, "\phi"] & X \sqcup f^C \arrow[d, "\xi"] \\
X' \arrow[r, "\psi_{X'}"] & X' \sqcup f^C
\end{tikzcd}$$

□

This construction is functorial in V, and as such, any PD-module W determines a functor,

$$\bullet \,\widehat{\otimes}\, W : \mathsf{FI}\text{-}\mathsf{Mod} \to \mathsf{FI}\text{-}\mathsf{Mod} \tag{5.4}$$

sending the FI-module V to the FI-module $V \,\widehat{\otimes}\, W$. If W is a finitely generated PD-module, then this funtor restricts to an endofunctor on the subcategory of finitely generated FI-modules. That is,

Theorem 5.4.3. *Let $V \in \mathsf{FI}\text{-}\mathsf{Mod}$ and $W \in \mathsf{PD}\text{-}\mathsf{Mod}$ both finitely generated. Then $V \mathbin{\widehat{\otimes}} W$ is a finitely generated FI-module.*

Proof. It suffices to show that the FI-module $\Lambda := M(S) \mathbin{\widehat{\otimes}} N(T, U)$ is finitely generated for arbitrary finite sets S, T and U. In particular, it suffices to show $\exists N \geq 0$ such that,

$$\Lambda = \operatorname{Span}(\Lambda_{\leq N}).$$

The vector space $\Lambda(Y)$ is a quotient of,

$$\bigoplus_{X \in \operatorname{ob}(\mathsf{FB})} M(S)_X \otimes N(T, U)_{X,Y},$$

and as such, a basis element $a \in \Lambda(Y)$ is determined by maps of the form,

$$\xi : S \hookrightarrow X, \qquad \alpha : T \hookrightarrow X, \qquad \beta : U \hookrightarrow Y, \qquad \gamma : \alpha^C \to \beta^C.$$

for a fixed set $X \in \operatorname{ob}(\mathsf{FB})$. Notice that ξ, α and β factor through maps,

$$S \hookrightarrow \operatorname{im}(\xi) \cup \operatorname{im}(\alpha), \qquad T \hookrightarrow \operatorname{im}(\xi) \cup \operatorname{im}(\alpha), \qquad U \hookrightarrow \operatorname{im}(\beta),$$

which themselves determine an element $z \in \Lambda(\operatorname{im}(\beta))$. Since $|\operatorname{im}(\beta)| \leq |U|$ we see that $a \in \operatorname{Span}(\Lambda_{\leq |U|})$, as desired.

$\square$

5.5 Representation stability in the context of families of $(\mathrm{S}_i, \mathrm{S}_n)$-bimodules

Let $\{W_{i,n} : i, n \in \mathbb{N}\}$ be a family of $(\mathrm{S}_i, \mathrm{S}_n)$-bimodules.

Definition 5.5.1. Fix $r \in \mathbb{Z}$ and partitions λ, μ. Let $c^{(r)}_{\lambda,\mu}(n) = c_{\lambda,\mu}(n)$ denote the multiplicity of the irreducible bimodule,

$$P(\lambda)_{n-r} \boxtimes P(\mu)_n,$$

in $W_{n-r,n}$.

Inspired in equal parts by the representation stability inherited by finitely generated FI-modules and by our observations from Part 4, we make the following definition.

Definition 5.5.2. (Representation stability in the context of $(\mathrm{S}_i, \mathrm{S}_n)$-bimodules) We say that the family $\{W_{i,n} : i, n \in \mathbb{N}\}$ of $(\mathrm{S}_i, \mathrm{S}_n)$-bimodules satsifies representation stability if, for all $r \in \mathbb{Z}$ and all partitions λ, μ, there exist constants N, C (depending on r, λ, μ) such that,

$$c^{(r)}_{\lambda,\mu}(n) = C,$$

for all $n \geq N$.

We say that a PD-module $W_{\bullet\bullet}$ satisfies representation stability if the associated family $\{W_{i,n} : i, n \in \mathbb{N}\}$ of $(\mathrm{S}_i, \mathrm{S}_n)$-bimodules does.

With this construction we are able to put some control on the underlying representation theory of finitely generated PD-modules in the form of the following stability statement.

Theorem 5.5.3. *Let W be a PD-module finitely generated in rank. Then W satisfies representation stability. That is, fix a rank $r \in \mathbb{Z}$. Let $c^{(r)}_{\lambda,\mu}(n)$ denote the multiplicity of,*

$$P(\lambda)_{n-r} \boxtimes P(\mu)_n$$

in $W_{n-r,r}$. Then there exist constants N, C such that,

$$c_{\mu,\lambda}(n) = C,$$

for all $n \geq N$.

Proof. Fix λ, μ partitions. The restriction to rank r, $W^{(r)}$ is a finitely generated PD-module. In Lemma 2.3.23 we introduced a finitely generated FI-module $P(\lambda)$ sending the finite set $\mathbf{n}$ to the irreducible S_n-module $P(\lambda)_n$ for n sufficiently large. Consider the FI-module,

$$P(\lambda) \mathbin{\widehat{\otimes}} W^{(r)}.$$

We start by computing the decomposition of,

$$(P(\lambda) \mathbin{\widehat{\otimes}} W^{(r)})_n,$$

into irreducible S_n-modules. Observe that $W^{(r)}_{a,n}$ is zero unless $a = n - r$. We therefore have that,

$$(P(\lambda) \mathbin{\widehat{\otimes}} W^{(r)})_n$$

is the quotient of,

$$P(\lambda)_{n-r} \otimes W_{n-r,n},$$

in which,

$$p \otimes \sigma \cdot w \qquad \text{is identified with} \qquad \sigma \cdot p \otimes w,$$

for all $p \in P(\lambda)_{n-r}, w \in W_{n-r,r}, \sigma \in \mathrm{S}_{n-r}$. This coincides with the definition of the tensor product over,

$$P(\lambda)_{n-r} \otimes_{\mathrm{S}_{n-r}} W_{n-r,n},$$

over the group ring $\Bbbk[\mathrm{S}_{n-r}]$. We write the $(\mathrm{S}_{n-r}, \mathrm{S}_n)$-bimodule $W_{n-r,n}$ as,

$$W_{n-r,n} = \bigoplus_{\nu,\mu} d_{\nu\eta}(n) P(\nu)_{n-r} \boxtimes P(\mu)_n,$$

where the sum is over partition ν, μ, and where $d_{\nu\mu}(n)$ is the multiplicity of $P(\nu)_{n-r} \boxtimes P(\mu)_n$ in $W_{n-r,n}$. We therefore have that,

$$\begin{aligned}
(P(\lambda) \mathbin{\widehat{\otimes}} W^{(r)})_n &\cong \bigoplus_{\nu,\mu} d_{\nu\mu}(n) P(\lambda)_{n-r} \otimes_{\mathrm{S}_{n-r}} (P(\nu)_{n-r} \boxtimes P(\mu)_n) \\
&\cong \bigoplus_{\nu,\mu} d_{\nu\mu}(n) \left(P(\lambda)_{n-r} \otimes_{\mathrm{S}_{n-r}} P(\nu)_{n-r}\right) \boxtimes P(\mu)_n \\
&\cong \bigoplus_{\lambda,\mu} c_{\lambda\mu}(n) P(\mu)_n,
\end{aligned}$$

where the last isomorphism follows from Schur's lemma (e.g., [12]), which implies that, for any partitions $\xi, \zeta \vdash a$,

$$P_\xi \otimes_{\mathrm{S}_a} P_\zeta \cong \begin{cases} \Bbbk & \xi = \zeta \\ 0 & \text{else} \end{cases}.$$

In other words, we have shown that the multiplicity with which $P(\mu)_n$ appears in,

$$(P(\lambda) \mathbin{\widehat{\otimes}} W^{(r)})_n,$$

is exactly the multiplicity with which $P(\lambda)_{n-r} \boxtimes P(\mu)_n$ appears in $W_{n-r,n}$. By Lemma 5.4.3, $P(\lambda) \mathbin{\widehat{\otimes}} W^{(r)}$ is a finitely generated FI-module. Hence by [7] Theorem 1.13, this multiplicity, $c_{\lambda\mu}(n)$, stabilizes for n sufficiently large.

□

Taking invariants of PD-modules. A special case of this operation is the reduction of a PD-module to an FI-module by taking invariants. Concretely,

Definition 5.5.4. Let $W \in \mathsf{PD\text{-}Mod}$. Define,

$$\mathcal{I}(W) := P(\varnothing) \mathbin{\widehat{\otimes}} W \in \mathsf{FI\text{-}Mod}.$$

It follows from the proof of Theorem 5.5.3 that the multiplicity of $P(\mu)_n$ in $\mathcal{I}(W^{(r)})_n$ is exactly the multiplicity with which,

$$P(\varnothing)_{n-r} \boxtimes P(\mu)_n \cong P_{(n-r)} \boxtimes P(\mu)_n,$$

appears in $W_{n-r,n}$. The following observation follows immediately.

Lemma 5.5.5. *Fix a rank $r \in \mathbb{Z}$ and let $W \in \mathsf{PD\text{-}Mod}$ be finitely generated in rank. Then $\mathcal{I}(W^{(r)})$ is a finitely generated FI-module satisfying,*

$$\mathcal{I}(W^{(r)})_n \cong \left(W^{(r)}_{n-r,n}\right)^{\mathrm{S}_{n-r}}.$$

5.6 Constructing PD-modules from S-modules

In this section we construct from an $\mathbf{S}$-module M a PD-module $\mathcal{W}(M)_{\bullet\bullet} \in \mathsf{PD}\text{-}\mathsf{Mod}$. Concretely, let $V, W \in \mathsf{Vect}$ and fix an $\mathbf{S}$-module M. Recall from Definition 2.1.8 the Schur functor,

$$\mathbb{S}_M(V) \in \mathsf{Vect},$$

and consider the (possibly infinite dimensional) vector space,

$$\mathbb{S}_M(V) \otimes W.$$

The symmetric algebra on this vector space naturally has the structure of a bimodule over $\mathrm{GL}(V) \times \mathrm{GL}(W)$. In particular, we have the decomposition (see [13]),

$$\begin{aligned}\mathrm{Sym}(\mathbb{S}_M(V) \otimes W) &\cong \bigoplus_{\mu} \mathbb{S}_\mu(\mathbb{S}_M(V)) \otimes \mathbb{S}_\mu(W)\\ &\cong \bigoplus_{\mu,\lambda} c^{(M)}_{\lambda\mu} \mathbb{S}_\lambda(V) \otimes \mathbb{S}_\mu(W),\end{aligned}$$

where the sum is over all pairs of partitions λ, μ, and where the multiplicities $c^{(M)}_{\lambda\mu} \in \mathbb{N}$ are the structure coefficients defined by,

$$\mathbb{S}_\mu(\mathbb{S}_M(V)) \cong \bigoplus_{\lambda} c^{(M)}_{\lambda\mu} \mathbb{S}_\lambda(V).$$

Remark 5.6.1. The coefficients $c^{(M)}_{\lambda\mu}$ are a special case of the plethysm problem described in Eq. (2.4).

Note that this determines a bigrading on $\mathrm{Sym}(\mathbb{S}_M(V) \otimes W)$ with,

$$\mathrm{Sym}(\mathbb{S}_M(V) \otimes W)_{i,n} := \bigoplus_{\substack{\mu \vdash i\\ \lambda \vdash n}} c^{(M)}_{\lambda\mu} \mathbb{S}_\lambda(V) \otimes \mathbb{S}_\mu(W),$$

for all $i, n \in \mathbb{N}$. This bigrading is seen to respect the algebra structure on,

$$\mathrm{Sym}(\mathbb{S}_M(V) \otimes W).$$

Definition 5.6.2. Let $\mathcal{W}(M) \in \mathsf{FB\text{-}BiMod}$ be the bimodule Schur-Weyl dual to,

$$\mathrm{Sym}(\mathbb{S}_M(V) \otimes W).$$

In particular, for any $i, n \in \mathbb{N}$, we have,

$$\mathcal{W}(M)_{i,n} := \bigoplus_{\substack{\mu \vdash i \\ \lambda \vdash n}} c^{(M)}_{\lambda\mu} P_\mu \boxtimes P_\lambda,$$

which determines a $(\mathrm{S}_i, \mathrm{S}_n)$-bimodule $\mathcal{W}(M)_{i,n}$.

Lemma 5.6.3. *Let M be an $\mathbf{S}$-module satisfying $M(1) = \Bbbk$. Then the FB-bimodule $\mathcal{W}(M)$ can be promoted to a PD-module.*

Proof. Fix $a \in \mathbb{N}$. Observe that an element $x \in \mathrm{Sym}(\mathbb{S}_M(V) \otimes W)_{a,a}$ determines a $\mathrm{GL}(V) \times \mathrm{GL}(W)$-equivariant map,

$$\mathrm{Sym}(\mathbb{S}_M(V) \otimes W)_{0,0} \xrightarrow{f_x} \mathrm{Sym}(\mathbb{S}_M(V) \otimes W)_{a,a},$$

by multiplication with x. Furthermore, for partitions $\lambda, \mu \vdash a$, the coefficient,

$$c^{(M)}_{\lambda\mu} = \delta_{\lambda\mu},$$

and so the bigraded component,

$$\mathrm{Sym}(\mathbb{S}_M(V) \otimes W)_{a,a} = \bigoplus_{\substack{\mu \vdash a \\ \lambda \vdash a}} c^{(M)}_{\lambda\mu} \mathbb{S}_\lambda(V) \otimes \mathbb{S}_\mu(W),$$

is Schur-Weyl dual to the group algebra,

$$\Bbbk[\mathrm{S}_a] \cong \bigoplus_{\mu \vdash a} P_\mu \boxtimes P_\mu.$$

Under this duality, any element $x \in \mathrm{Sym}(\mathbb{S}_M(V) \otimes W)_{a,a}$ is of the form,

$$(v_1 \cdots v_a \otimes w_1 \cdots w_a) \otimes y \in \left(V^{\otimes a} \otimes W^{\otimes a}\right) \otimes_{\mathrm{S}_a} \Bbbk[\mathrm{S}_a],$$

with $v_i \in V$ and $w_j \in W$ and $y \in \Bbbk[\mathrm{S}_a]$, and the map f_x is given by,

$$\begin{aligned} f_x : \Bbbk &\to \left(V^{\otimes a} \otimes W^{\otimes a}\right) \otimes \Bbbk[\mathrm{S}_a] \\ b &\mapsto b(v_1 \cdots v_a \otimes w_1 \cdots w_a) \otimes y \end{aligned}$$

In particular, any element $\sigma \in \mathrm{S}_a$ determines an element,

$$x_\sigma = (1 \cdots 1 \otimes 1 \cdots 1) \otimes \sigma \in \left(V^{\otimes a} \otimes W^{\otimes a}\right) \otimes \Bbbk[\mathrm{S}_a],$$

and thus a map f_{x_σ}, which we denote simply f_σ. Note that multiplication by x_σ determines maps,

$$(f_\sigma)_{i,n} : \mathrm{Sym}(\mathbb{S}_M(V) \otimes W)_{i,n} \to \mathrm{Sym}(\mathbb{S}_M(V) \otimes W)_{i+a,n+a},$$

for all $i, n \in \mathbb{N}$.

Any morphism $\Delta \in \mathrm{Hom}_{\mathsf{PD}}((0,0),(a,a))$ is equivalent to an element $\sigma \in \mathrm{S}_a$. Denote the morphism corresponding to $\sigma \in \mathrm{S}_a$ by Δ_σ. To Δ_σ we assign the map,

$$W_{0,0} \to W_{a,a},$$

induced by f_σ. Furthermore, for any morphism,

$$\Delta = (\alpha, \beta, \gamma) \in \mathrm{Hom}_{\mathsf{PD}}((i,n),(i+a,n+a)),$$

there is a canonical identification of γ with an element of $\sigma \in \mathrm{S}_a$. To such a morphism we associate the map,

$$(f_\sigma)_{i,n} : \mathrm{Sym}(\mathbb{S}_M(V) \otimes W)_{i,n} \to \mathrm{Sym}(\mathbb{S}_M(V) \otimes W)_{i+a,n+a},$$

which induces a map $\mathcal{W}(M)_{i,n} \to \mathcal{W}(M)_{i+a,n+a}$ as desired.

$\square$

Theorem 5.6.4. *Let M be an $\mathbf{S}$-module such that $M(1) = \Bbbk$. The PD-module $\mathcal{W}(M)$ is finitely generated in rank.*

Proof. Fix $k > 0$ and partitions λ, μ such that $|\lambda| = |\mu| + k$. We have that,

$$\mathrm{Sym}(\mathbb{S}_M(V) \otimes W) \cong \bigoplus_{\mu} \mathbb{S}_\mu(\mathbb{S}_M(V)) \otimes \mathbb{S}_\mu(W).$$

It therefore suffices to consider the decomposition,

$$\mathbb{S}_\mu(\mathbb{S}_M(V)) \cong \bigoplus_{\lambda} c^{(M)}_{\lambda\mu} \mathbb{S}_\lambda(V),$$

and in particular, we restrict out attention to those partitions λ such that $|\lambda| = |\mu| + k$. In Chapter 4 we detailed this decomposition for the case $M = \mathsf{Lie}$, and the same reasoning applies here. Concretely, for a fixed partition μ, the subspace,

$$\bigoplus_{\lambda : |\lambda| = |\mu| + k} c^{(M)}_{\lambda\mu} \mathbb{S}_\lambda(V)$$

is determined by all ways to decompose μ into a good μ-decomposition $\mu_1, \ldots, \mu_j$ such that λ appears with positive multiplicity in the assembly (see Definition 4.2.14),

$$(\mu_1, \ldots, \mu_j) \smile (m_{i_1}, \ldots, m_{i_j}),$$

where m_{i_*} are distinct partitions appearing in the $\mathbb{S}$-module M.

For $|\mu|$ sufficiently large, any assembly,

$$(\mu_1, \ldots, \mu_j) \smile (m_{i_1}, \ldots, m_{i_j}),$$

of target size $|\mu| + k$ is forced to have $|\mu_1| \geq |\mu| - k$ and $m_{i_1} = m_1 = \Bbbk$.

For $|\mu| > k$, any assembly of target size $|\mu| + k$ must involve m_1. Indeed, once $|\mu| \geq k+1$, the smallest target size possible of an assembly that does not involve m_1 is

$2\cdot(k+1) > |\mu|+k$. Without loss of generality, then, say that μ_1 is paired with $m_1 = \Bbbk$. Observe that $|\mu_1| \geq |\mu| - k$. Indeed, suppose for a contradiction that $|\mu| = |\mu| - l$ with $l > k$. Then the target size of the resulting assembly is at least,

$$|\mu_1| + (|\mu| - |\mu_1|) \cdot 2 = 2 \cdot l > |\mu| + k,$$

contradicting the assumption that the assembly is of size $|\mu| + k$. It follows that any contribution to the (λ, μ)-decomposition puzzle associated to M is obtained from a contribution to a $(\lambda_\circ, \mu_\circ)$-decomposition puzzle where $|\mu_\circ| \leq k$ by taking the induction product with a partition of size $|\mu| - |\mu_\circ|$. That is to say, the subspace,

$$\bigoplus_{\lambda:|\lambda|=|\mu|+k} c_{\lambda\mu}^{(M)} \mathbb{S}_\lambda(V)$$

lies in the PD-span of the subpace,

$$\bigoplus_{\substack{\lambda:|\lambda|=|\mu|+k \\ |\mu|\leq k}} c_{\lambda\mu}^{(M)} \mathbb{S}_\lambda(V).$$

We have therefore shown that the PD-module is finitely generated in rank. □

5.6.1 Application to the structure coefficients $c_{\lambda\mu}$

Recall the definition of the coefficients $c_{\lambda\mu}$ from Chapter 4,

$$\mathrm{Sym}(\mathcal{L}(V) \otimes \Bbbk^n) \cong \bigoplus_{\lambda,\mu} c_{\lambda\mu} \mathbb{S}_\lambda V \otimes \mathbb{S}_\mu(\Bbbk^n).$$

It follows from Definition 2.1.11 that the coefficients $c_{\lambda\mu}$ are exactly the coefficients $c_{\lambda\mu}^{(\mathsf{Lie})}$ arising from the PD-module,

$$\mathcal{W}(\mathsf{Lie}),$$

with $W = \Bbbk^n$. In particular, it is an immediate corollary to Theorem 5.6.4 that the coefficients $c_{\lambda\mu}$ satisfy representation stability as stated in Definition 5.5.2.

Theorem 5.6.5. *The coefficients $c_{\lambda\mu}$ satisfy representation stability.*

Remark 5.6.6. This result completes the story started in Chapter 4 of the coefficients $c_{\lambda\mu}$. In that chapter we provided evidence, in the form of large amounts of computation and some visualizations, that there was a generalization of the representation stability of Church-Farb [9] to a two-dimensional setting. In particular the visualizations provided patterns suggestive of some stability phenomenon, and the theorem above confirms that these patterns were a consequence of some latent general structure.

5.7 The category PDI

The category PD was obtained from the product category $\mathsf{FB} \times \mathsf{FB}$ by the addition of so-called diagonal maps (see Definition 5.1.1). It turns out that these diagonal maps interact nicely with the larger product category $\mathsf{FI}^{\mathsf{op}} \times \mathsf{FI}$, giving rise to the larger category PDI, which we now define.

Definition 5.7.1. Let PDI be the category with

1. Objects. Pairs of finite sets (X, Y).
2. Morphisms. Generated by

   (a) FI-morphisms. A map $(X, Y) \to (X', Y)$ for a map in $\mathrm{Hom}_{\mathsf{FI}}(X, X')$.

   (b) $\mathsf{FI}^{\mathsf{op}}$-morphisms. A map $(X, Y') \to (X, Y)$ for a map $\mathrm{Hom}_{\mathsf{FI}^{\mathsf{op}}}(Y', Y)$.

   (c) PD-morphisms. A map $(X, Y) \to (X', Y')$ for a map in $\mathrm{Hom}_{\mathsf{PD}}((X, Y), (X', Y'))$.

   subject to the following compatibility condition. Given,

$$\Delta = (\alpha, \beta, \gamma) \in \mathrm{Hom}_{\mathsf{PD}}((X, Y), (X', Y')),$$

we have that $\alpha \in \mathrm{Hom}_{\mathsf{FI}}(X, X')$ and $\beta \in \mathrm{Hom}_{\mathsf{FI}^{\mathsf{op}}}(Y', Y)$. The following diagram commutes for any choice of bijection γ.

$$\begin{tikzcd} & (X', Y') \arrow[d, "\beta"] \\ (X, Y) \arrow[ur, "\Delta=(\alpha,\beta,\gamma)"] \arrow[r, "\alpha"] & (X', Y) \end{tikzcd}$$

We give the following simple characterization of morphisms in PDI.

Lemma 5.7.2. *A morphism $(X, Y) \to (X', Y')$ in PDI determines, and is determined by, the following data.*

1. *An injection $f : X \hookrightarrow X'$, and*
2. *A injection $g : Y' \hookrightarrow Y \sqcup f^C$.*

Proof. We first use f to build a diagonal morphism $(X, Y) \to (X', Y'')$ where $Y'' = Y \sqcup \overline{f}$. Notice that there is a natural inclusion $\iota : Y \hookrightarrow Y''$ with $\overline{\iota} = \overline{f}$. Such a morphism is obtained from the triple of data,

$$(f : X \hookrightarrow X', \iota : Y \hookrightarrow Y'', \gamma = id : \overline{\iota} \simeq \overline{f}).$$

Then g defines an $\mathsf{FI}^{\mathsf{op}}$-morphism $(X', Y'') \to (X', Y')$.

Conversely, given a diagonal morphism $(\alpha, \beta, \gamma) : (X, Y) \to (X'', Y')$ and an $\mathsf{FI}^{\mathsf{op}}$-morphism $g : (X', Y'') \to (X', Y')$ whose composition is a map $(X, Y) \to (X', Y')$ we first set $f = \alpha : X \hookrightarrow X'$. Then we can use $\beta^{-1} \sqcup \gamma$ to identify $Y'' = \beta(Y) \sqcup \overline{\beta}$ with $Y \sqcup \overline{f}$ whence the map g can be viewed as a map $Y' \hookrightarrow Y \sqcup \overline{f}$, as desired. $\square$

As a simple application of this lemma we have the following characterization of morphisms in PDI. This should be compared with Lemma 5.1.4.

Lemma 5.7.3. *(Characterization of morphisms in* PDI*.)*

1. $\mathrm{Hom}_{\mathsf{PDI}}((\varnothing,\varnothing),(X,Y)) = \mathrm{Hom}_{\mathsf{FI}}(Y,X) = \mathrm{Hom}_{\mathsf{FI}^{\mathsf{op}}}(X,Y)$,

2. $\mathrm{Hom}_{\mathsf{PDI}}((S,\varnothing),(X,Y)) = \mathrm{Hom}_{\mathsf{FI}}(Y \sqcup S, X) = \mathrm{Hom}_{\mathsf{FI}^{\mathsf{op}}}(X, Y \sqcup S)$,

3. $\mathrm{Hom}_{\mathsf{PDI}}((\varnothing,T),(X,Y)) = \mathrm{Hom}_{\mathsf{FI}}(Y, X \sqcup T) = \mathrm{Hom}_{\mathsf{FI}^{\mathsf{op}}}(X \sqcup T, Y)$,

Definition 5.7.4. (PDI-modules) A PDI-module is a functor $\mathsf{PDI} \to \mathsf{Vect}$. We denote the category of PDI-modules by $\mathsf{PDI}\text{-}\mathsf{Mod}$.

Recall from the representable functors construction in Definition 2.2.9 that there is a standard way to produce PDI-modules from objects $(S,T) \in \mathrm{ob}(\mathsf{PDI})$.

Definition 5.7.5. Fix a pair of finite sets $(S,T) \in \mathrm{ob}(\mathsf{PDI})$. Define,

$$W_{(S,T)} := \mathsf{R}_{\mathsf{PDI}}((S,T), \bullet\bullet) \in \mathsf{PDI}\text{-}\mathsf{Mod},$$

Concretely, $W_{(S,T)}$ sends the pair of finite sets $(X,Y) \in \mathrm{ob}(\mathsf{PDI})$ to the free $\Bbbk$-vector space

$$\Bbbk[\mathrm{Hom}_{\mathsf{PDI}}((S,T),(X,Y))].$$

PDI-morphisms naturally act on the basis by post-composition.

The following examples follow immediately from Lemma 5.7.3.

Example 5.7.6.

1. $W_{(\varnothing,\varnothing)}(X,Y) = \Bbbk[\mathrm{Hom}_{\mathsf{FI}}(Y,X)]$

2. $W_{(S,\varnothing)}(X,Y) = \Bbbk[\mathrm{Hom}_{\mathsf{FI}}(Y \sqcup S, X)]$

3. $W_{(\varnothing,T)}(X,Y) = \Bbbk[\mathrm{Hom}_{\mathsf{FI}}(Y, X \sqcup T)]$

5.8 Endofunctors on FI-Mod arising from PDI-modules

We unpack Definition 2.2.13 in the case $\mathsf{C} = \mathsf{FI}$.

Definition 5.8.1. (Tensor product over FI) Given $V \in \mathsf{FI}\text{-}\mathsf{Mod}$ and $W \in \mathsf{FI}^{\mathsf{op}}\text{-}\mathsf{Mod}$ the tensor product over FI,

$$V \otimes_{\mathsf{FI}} W \in \mathsf{Vect},$$

is defined by,

$$V \otimes_{\mathsf{FI}} W = \left(\bigoplus_{Y \in Ob(\mathsf{FI})} V(Y) \otimes W(Y) \right) / \langle f_*(u_X) \otimes v_Y \equiv u_S \otimes f^*(v_Y) : f : X \hookrightarrow Y \rangle,$$

or equivalently by,

$$V \otimes_{\mathsf{FI}} W = \left(\bigoplus_{n \geq 0} V_n \otimes_{\mathrm{S}_n} W_n \right) / \langle f_*(u_n) \otimes v_{n+1} \equiv u_n \otimes f^*(v_{n+1}) : f : [n] \hookrightarrow [n+1] \rangle.$$

Remark 5.8.2. If $W \in \mathsf{FI}\text{-}\mathsf{BiMod}$ then $V \otimes_{\mathsf{FI}} W \in \mathsf{FI}\text{-}\mathsf{Mod}$. Indeed, given a finite set X, then $W(-, X) \in \mathsf{FI}^{\mathsf{op}}\text{-}\mathsf{Mod}$ and

$$(V \otimes_{\mathsf{FI}} W)(X) := V \otimes_{\mathsf{FI}} W(-, X) \in \mathsf{Vect} \tag{5.5}$$

Furthermore this evaluation is readily seen to be functorial. This gives us a recipe for creating functors $\mathsf{FI}\text{-}\mathsf{Mod} \to \mathsf{FI}\text{-}\mathsf{Mod}$ from PDI-modules. Concretely, given $W \in \mathsf{PDI}\text{-}\mathsf{Mod}$ we obtain a functor $- \otimes_{\mathsf{FI}} W : \mathsf{FI}\text{-}\mathsf{Mod} \to \mathsf{FI}\text{-}\mathsf{Mod}$ from (5.5) by considering W as an FI-bimodule.

Example 5.8.3.

1. Consider $W_{(\varnothing,\varnothing)} \in \mathsf{PDI}\text{-}\mathsf{Mod}$. Given a finite set X the evaluation,

   $$W_{(\varnothing,\varnothing)}(-, X) = \Bbbk[\mathrm{Hom}_{\mathsf{FI}^{\mathsf{op}}}(X, -)],$$

   is the representable functor $\mathsf{R}_{\mathsf{FI}^{\mathsf{op}}}(X, \bullet)$. The tensor product,

   $$V \otimes_{\mathsf{FI}} \mathsf{R}_{\mathsf{FI}^{\mathsf{op}}}(X, \bullet) \cong V(X),$$

by the Yoneda Lemma (Lemma 2.2.12). It follows from the construction that,

$$V \otimes_{\mathsf{FI}} W_{(\varnothing,\varnothing)} \cong V.$$

2. By considering,

$$W_{(S,\varnothing)}(-,X) = \Bbbk[\mathrm{Hom}_{\mathsf{FI}^{\mathsf{op}}}(X, \bullet \sqcup S)],$$

a similar argument to that above gives that,

$$V \otimes_{\mathsf{FI}} W_{(S,\varnothing)} \cong V(\bullet \sqcup S).$$

5.9 Free PDI-modules

In this section we present another application of the free object paradigm (Section 2.2.6), this time in the context of the inclusion of categories,

$$\mathsf{FB} \times \mathsf{FB} \hookrightarrow \mathsf{PDI}.$$

Following Definition 2.2.19 we have a restriction functor,

$$\mathrm{Res}^{\mathsf{PDI}}_{\mathsf{FB}\times\mathsf{FB}} : \mathsf{PDI\text{-}Mod} \to \mathsf{FB\text{-}BiMod},$$

defined by precomposition with the inclusion. By a similar argument to Lemma 5.2.3 we have that the left adjoint,

$$\mathrm{Ind}^{\mathsf{PDI}}_{\mathsf{FB}\times\mathsf{FB}} : \mathsf{FB\text{-}BiMod} \to \mathsf{PDI\text{-}Mod}$$

satisfies,

$$\mathrm{Ind}^{\mathsf{PDI}}_{\mathsf{FB}\times\mathsf{FB}}(W)_{i,n-k} \cong \bigoplus_{m\in\mathbb{N}} \left(\mathrm{Ind}^{S(i,n)}_{S(i-m,n-m)\times S(m,m)} W_{i-m,n-m} \boxtimes \Bbbk[\mathrm{S}_m]\right)^{1\times \mathrm{S}_k} \tag{5.6}$$

Remark 5.9.1. We omit the proof of the above for two reasons. First, as was already mentioned, the proof of Lemma 5.2.3 is very similar. Second, this statement is not used anywhere beyond the current section.

Example 5.9.2. Let A be the FB-bimodule supported in bidegree $(2,0)$ by the $(\mathrm{S}_2, \mathrm{S}_0)$-module,

$$A_{2,0} = \ydiagram{1,1}.$$

In Fig. 5.3 we show the decomposition into irreducible bimodules of $\mathrm{Ind}^{\mathsf{PDI}}_{\mathsf{FB}\times\mathsf{FB}}(A)$ in small bidegree.

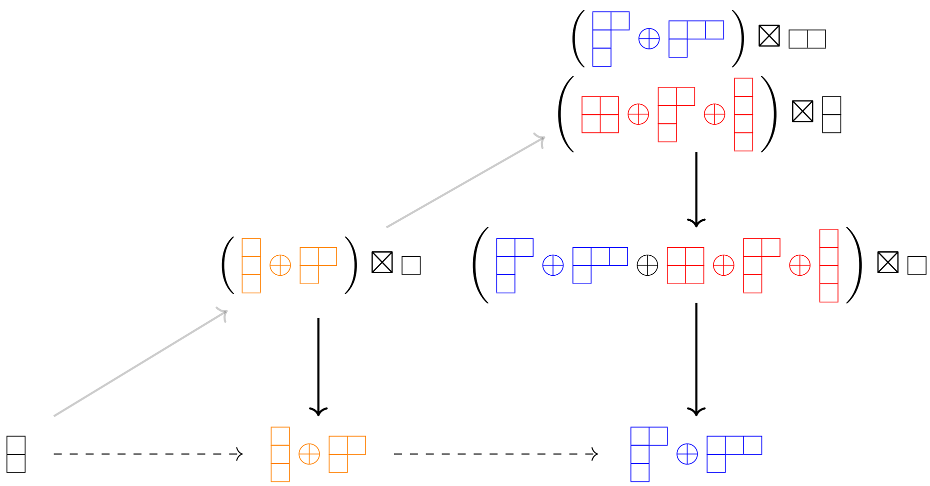


Figure 5.3: The diagonal lines represent the induction to at PD-module. The bold vertical lines represent the induction from a PD-module to a PDI-module by applying Eq. (5.6). The dashed lines show an FI-module sitting in the 0-th row.

Notice in the example above that the PD-module $\mathrm{Ind}^{\mathsf{PD}}_{\mathsf{FB}\times\mathsf{FB}}(B)$ is sitting along the top diagonal. This is a general feature of the construction, and can be seen directly from Eq. 5.6 by setting $k = 0$. Furthermore, given that the inclusion,

$$\mathsf{FB} \times \mathsf{FB} \hookrightarrow \mathsf{PDI},$$

factors through the inclusion,

$$\mathsf{FB} \times \mathsf{FB} \hookrightarrow \mathsf{PD},$$

we have that the functor $\mathrm{Ind}^{\mathsf{PDI}}_{\mathsf{FB}\times\mathsf{FB}}(\bullet)$ factors through $\mathsf{PD}\text{-}\mathsf{Mod}$,

$$\begin{tikzcd} \mathsf{FB\text{-}BiMod} \arrow[rr, "\mathrm{Ind}_{\mathsf{FB}\times\mathsf{FB}}^{\mathsf{PDI}}(\bullet)"] \arrow[dr, "\mathrm{Ind}_{\mathsf{FB}\times\mathsf{FB}}^{\mathsf{PD}}(\bullet)"'] & & \mathsf{PDI\text{-}Mod} \\ & \mathsf{PD\text{-}Mod} \arrow[ur, "\mathrm{Ind}_{\mathsf{PD}}^{\mathsf{PDI}}(\bullet)"'] & \end{tikzcd}$$

By uniqueness of adjoints, we can describe the free functor,

$$\mathrm{Ind}_{\mathsf{PD}}^{\mathsf{PDI}} : \mathsf{PD\text{-}Mod} \to \mathsf{PDI\text{-}Mod},$$

associated to the natural inclusion of categories,

$$\mathsf{PD} \hookrightarrow \mathsf{PDI}.$$

In particular, we have that,

$$\mathrm{Ind}_{\mathsf{PD}}^{\mathsf{PDI}}(W)_{i,n} = \bigoplus_{k \geq 0} (W_{i,n+k})^{(1 \times \mathrm{S}_k)} .$$

Remark 5.9.3. Notice that there is a forgetful functor,

$$\mathcal{F}_1 : \mathsf{PDI\text{-}Mod} \to \mathsf{FI\text{-}Mod},$$

by restricting to objects of the form $(X, \varnothing)$. We use this to construct a functor,

$$\mathcal{F}_2 : \mathsf{PD\text{-}Mod} \to \mathsf{FI\text{-}Mod},$$

defined as the composition $\mathcal{F}_1 \circ \mathrm{Ind}_{\mathsf{PD}}^{\mathsf{PDI}}$. Given a PD-module W, we have that,

$$\mathcal{F}_2(W)_i \cong \bigoplus_{k \geq 0} (W_{i,k})^{(1 \times \mathrm{S}_k)}.$$

That is, $\mathcal{F}_2$ is precisely the invariants functor constructed in Definition 5.5.4.

CHAPTER 6

EXTENDED WHITNEY HOMOLOGY

6.1 Whitney homology of the lattice of set partitions

We recall definitions of the order complex $\Delta(P)$ of a finite graded poset P, and its Whitney homology. Much of this material is standard. See, for example, Stanley [31], or Aguiar and Mahajan [2].

6.1.1 Order homology of a poset

Fix a ground field $\Bbbk$. Let P be a finite poset with unique minimal element $\perp \in P$ and unique maximal element $\top \in P$. A *maximal chain* in P is a totally ordered subset which is maximal under inclusion. Say that P is *graded* if all its maximal chains have the same length. If P is graded then, for $x \in P$, define $\operatorname{rank}(x)$ to be the length r of a(ny) maximal chain,

$$\perp < x_1 < \cdots < x_r < x,$$

from $\perp$ to x. We say that the *rank* of P is $\operatorname{rank}(\top)$.

Remark 6.1.1. There is a definition of rank presented in the theory of PD-modules. It turns out that in our context, these two otherwise distinct notions coincide. In addition, there can be no ambiguity over the rank to which we refer, and as such we do not attempt to distinguish them in the nomenclature.

Homology of a poset. The strict chains in P between $\perp$ and $\top$ have the structure of a simplicial complex, called the order complex $\Delta(P)$ of P. Concretely, suppose P is of rank r. For $-1 \leq j \leq r-2$, the chain group $\mathcal{C}_j(P)$ is the vector space with basis consisting of strict chains,

$$\perp < x_1 < \cdots < x_{j+1} < \top,$$

of length $j+1$. Otherwise $\mathcal{C}_j(P)$ is zero. Note that $\mathcal{C}_{-1}(P)$ is one dimensional and spanned by the chain $\perp < \top$. The differential ∂_j is defined via the formula,

$$\partial_j(\perp < x_1 < \cdots < x_{j+1} < \top) = \sum_{l=1}^{j+1} (-1)^l (\perp < x_1 < \cdots < \hat{x}_l < \cdots < x_{j+1} < \top),$$

where $\hat{x}_l$ denoted the omission of x_l from the chain. The order complex $\Delta(P)$ of P is the simplicial complex $(C_*(P), \partial)$. Write $H_*(P)$ for the homology of this complex.

Homology of an interval. For elements $x < y \in P$, define the interval $(x, y) = \{z \in P : x < z < y\}$. We can make a similar definition for the order complex $\Delta(x, y)$ of the interval (x, y) as the simplicial complex arising from the strict chains in P between x and y. Namely, a basis for the j-th chain group $\mathcal{C}_j(x, y)$ of $\Delta(x, y)$ consists of strict chains,

$$x < z_1 < \cdots < z_{j+1} < y.$$

Similarly we have that $\mathcal{C}_{-1}(x, y)$ is one dimensional and spanned by $x < y$, and the differential ∂ is defined exactly as above. The homology $H_*(x, y)$ of the interval (x, y) is the homology of this complex. It is not hard to see that $\dim(x, y) = \operatorname{rank}(y) - \operatorname{rank}(x) - 2$.

Remark 6.1.2. A poset P is *Cohen-Macaulay* if every interval (x, y) in P has homology concentrated in top degree. That is,

$$H_j(x, y) = 0,$$

for all $j < \dim(x, y)$. It is well-known (see, for example, [30]) that the property of being Cohen-Macaulay is inherited by rank-selected sub-posets, i.e., by posets,

$$P^S = \{x \in P : \mathrm{rank}(x) \in S\},$$

where $S \subset \{0, \ldots, \mathrm{rank}(P)\}$ is a subset of the possible ranks of the graded, Cohen-Macaulay poset P.

In the upcoming definition of Whitney homology, which is standard, we introduce a non-standard grading. For that purpose we first define, for $x \in P$, the *length* of x as,

$$l(x) = \mathrm{rank}(P) - \mathrm{rank}(x) + 1. \tag{6.1}$$

We are now ready to present the definition of Whitney homology, with this modified grading.

Definition 6.1.3. (Whitney homology) The i-th Whitney homology $\mathcal{WH}_i(P)$ of P is the direct sum,

$$\mathcal{WH}_i(P) = \bigoplus_{\substack{x \in P \\ l(x) = i}} H_*(\perp, x),$$

of interval homologies $H_*(\perp, x)$ for $x \in P$ of length i.

Remark 6.1.4. The usual grading is over x of rank i. Our modification is intended to simplify notation down the line.

Remark 6.1.5. Let G be a group of poset automorphisms of P. Then the poset homology $H_*(P)$ of P is a G-module [30]. Similarly, if G is a group of poset automorphisms of an interval (x, y) then the interval homology,

$$H_*(x, y),$$

of (x, y) is a G-module. Understanding the G-module structure of certain interval and poset homologies is a central theme of this work, specifically in the context of the lattice of set partitions, which we now recall.

6.1.2 The lattice of set partitions

Definition 6.1.6. Fix $n \in \mathbb{N}$. A set partition $x = \{B_1|\dots|B_i\}$ of $\{1, 2, \dots, n\}$ is a decomposition,

$$\{1, 2, \dots, n\} = \bigsqcup_{j=1}^{i} B_j,$$

into disjoint sets B_j called *blocks* of x. The set, Π_n, of all such blocks naturally admits the structure of a poset, ordered by refinement. Concretely, $x \leq y$ if each block of y if a union of blocks of x. The lattice of set partitions Π_n has unique maximum element and minimum elements,

$$\top = \{1, 2, \cdots, n\},$$
$$\perp = \{1|2|\cdots|n\}.$$

We define the length and type of a set partition. These notions provide natural decompositions of the lattice Π_n.

Remark 6.1.7. We use the symbol Π_n to refer to the lattice of set partitions. To avoid ambiguity, the symbol $\prod$ is used to denote the cartesian product.

Definition 6.1.8. Fix $n \in \mathbb{N}$ and consider a set partition $x = (B_1, \dots, B_i) \in \Pi_n$.

1. We define[1] the *length* of x, denoted $l(x)$, to be i. Further, let $\Pi_{i,n}$ denote the subset of Π_n consisting of set partitions of length i:

$$\Pi_{i,n} = \{x \in \Pi_n : l(x) = i\}.$$

We have the following decomposition $\Pi_n = \bigsqcup_{i=1}^{n} \Pi_{i,n}$ by lengths.

[1] In fact, we redefine length. Note that this definition agrees with the generic definition given in Eq. 6.1.

2. We associate a partition $\mathrm{type}(x) \vdash n$, called the *type* of x, determined by the sizes of the blocks in x. Concretely, $\mathrm{type}(x) = (|B_1|, \ldots, |B_i|) \vdash n$.

Further, let Π_λ denote the subset of Π_n consisting of set partitions of type λ:

$$\Pi_\lambda = \{x \in \Pi_n : \mathrm{type}(x) = \lambda\}.$$

We have the following decomposition,

$$\Pi_n = \bigsqcup_{\lambda \vdash n} \Pi_\lambda,$$

by types.

Example. (Lattice of set partitions Π_3)

Consider the set partition $x = \{12|3\} \in \Pi_3$. It has blocks $B_1 = \{1, 2\}$, $B_2 = \{3\}$ and $\mathrm{type}(x) = (2, 1) \vdash 3$. This fits into the full lattice of set partitions Π_3 as follows.

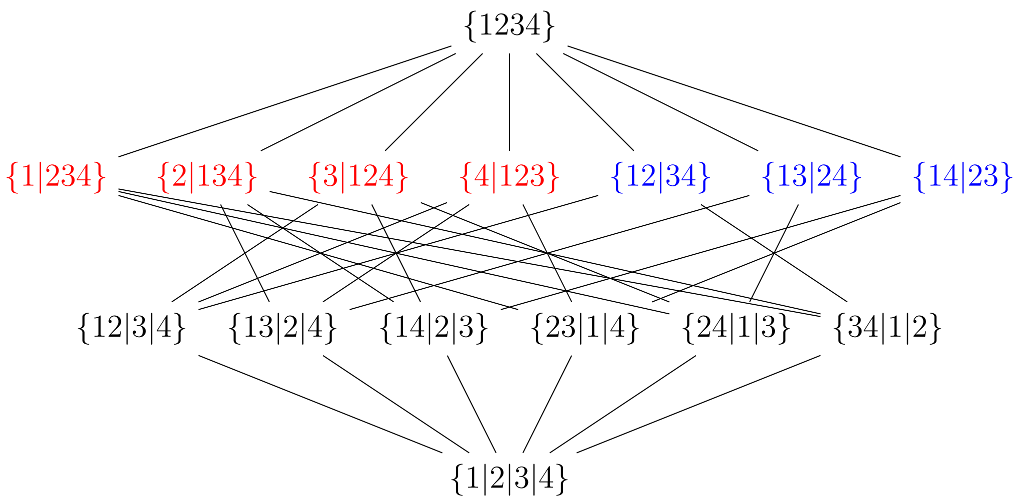
Type	Π_3			Rank	Length
$(3) \vdash 3$		$\{123\}$		2	1
$(2,1) \vdash 3$	$\{12\|3\}$	$\{13\|2\}$	$\{23\|1\}$	1	2
$(1,1,1) \vdash 3$		$\{1\|2\|3\}$		0	3

Notice that the rank and the length determine one another, and that elements $x \in \Pi_3$ of the same type are also of the same rank and length. These will be seen to be general features of the type, rank and length of a set partition (see Lemma 6.1.9).

Example. (Lattice of set partitions Π_4)

For $n = 4$ the poset structure is somewhat more complex. We first present the poset diagram for Π_4:

Π_4



The type, rank and length are summarized below:

Type	Rank	Length
(4)	3	1
$(3,1)$ $(2,2)$	2	2
$(2,1,1)$	1	3
$(1,1,1,1)$	0	4

Rank determines length. The type, rank and length of a set partition are related to one another by the following simple observation which we present without proof.

Lemma 6.1.9. *Fix $n \in \mathbb{N}$ and let $x \in \Pi_n$. The rank of x and the length of x determine one another:*

$$\operatorname{rank}(x) = n - l(x).$$

Moreover, both are determined by the type $\lambda = (\lambda_1, \ldots, \lambda_i) \vdash n$ of x:

$$\operatorname{rank}(x) = |\lambda| - i \qquad l(x) = i.$$

It is also immediate to see that the rank of Π_n is $n - 1$.

Product decomposition of the interval $(\perp, x)$**.** It is well-known (see [32], for example) that the interval $(\perp, x)$ admits a poset decomposition in terms of smaller posets Π_a.

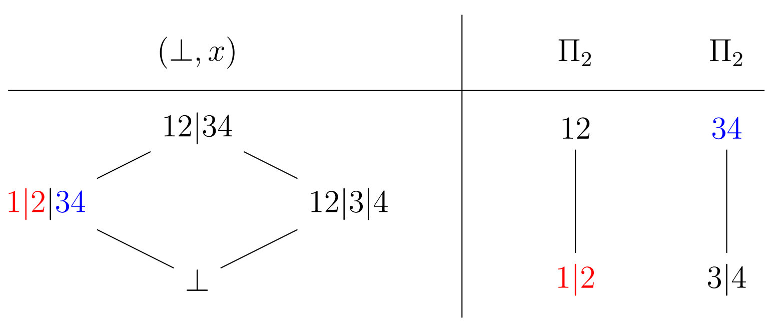
Lemma 6.1.10. *Let* $x \in \Pi_n$ *of type* $\lambda = 1^{m_1}2^{m_2}\cdots \vdash n$. *Then there is a poset isomorphism,*

$$(\perp, x) \cong \prod_j \left(\Pi_j^{\times m_j}\right).$$

Proof. Write $x = (B_1, \ldots, B_i)$. Any set partition $z \in (\perp, x)$ satisfies $z \leq x$, and as such consists of blocks that refine the blocks of x. So the block $B_k \in x$ of size j is refined into blocks $B_{k1}, \ldots, B_{kl_k} \in z$. This refinement determines an element of $(\perp, B_k) \cong \Pi_j$. Conversely, given elements $(B_{k1}|\cdots|B_{kl_k}) \in (\perp, B_k) \cong \Pi_j$ for each block of x we obtain an element $z \in (\perp, x)$ by concatenation. $\square$

We demonstrate this result with a simple example.

Example 6.1.11. Consider the set partition $x = 12|34$ of type $(2, 2)$ in Π_4. We can represent the poset $(\perp, x)$ via its poset diagram. Additionally we represent $(\perp, 12)$ and $(\perp, 34)$ via thier poset diagrams, each of which is isomorphic to Π_2.



For example, the isomorphism $(\perp, x) \cong \Pi_2 \times \Pi_2$ sends $1|2|34 \in (\perp, x)$ to $(1|2, 34) \in \Pi_2 \times \Pi_2$.

6.1.3 Whitney complexes and Whitney homology

In this section we apply the general machinery described in Definition 6.1.3 to the case $P = \Pi_n$. Three threads will be running in tandem:

1. We consider the entire poset Π_n.
2. We consider the type-selected subsets Π_λ for $\lambda \vdash n$.
3. We consider the length-selected subsets $\Pi_{i,n}$ for $i < n$.

Definition 6.1.12. Fix $n \in \mathbb{N}$. For $x \in \Pi_n$, let $\mathcal{C}_x$ denote the order complex $\Delta(\perp, x)$.

1. For $\lambda \vdash n$, let,
$$\mathcal{C}_\lambda := \bigoplus_{x \in \Pi_\lambda} \mathcal{C}_x. \tag{6.2}$$
We call $\mathcal{C}_\lambda$ the type-selected order complex of Π_n (of type λ). The Whitney homology of type λ, $\mathcal{WH}_\lambda$, is the homology of the complex $\mathcal{C}_\lambda$.

2. We further group these complexes by length, defining the length-selected order complex $\mathcal{C}_{i,n}$ as,
$$\mathcal{C}_{i,n} := \bigoplus_{\substack{\lambda \vdash n \\ l(\lambda)=i}} \mathcal{C}_\lambda = \bigoplus_{x \in \Pi_{i,n}} \mathcal{C}_x. \tag{6.3}$$
The i-th Whitney homology, $\mathcal{WH}_{i,n}$, is the homology of the complex $\mathcal{C}_{i,n}$.

Remark 6.1.13.

1. By Lemma 6.1.9 and Remark 6.1.2 we see that the homology of $\mathcal{C}_\lambda$ and $\mathcal{C}_{i,n}$ are concentrated in top-degree. We therefore use $\mathcal{WH}_\lambda$ and $\mathcal{WH}_{i,n}$ to refer to homology in this top degree.

2. In the literature (e.g., [1], [18]) it is more common to define the *rank-selected* order complex $\mathcal{C}_{i,n}$ (and correspondingly, the rank-selected Whitney homology $\mathcal{WH}_{i,n}$) as summing over terms of rank i, but in our context, this modified grading will be

more succinct. Of course, by Lemma 6.1.9, fixing the length $l(x) = i$ is equivalent to fixing the rank $\mathrm{rank}(x) = n - i$, so this change of grading is only cosmetic.

Example 6.1.14. Consider the set partition $x = \{1|234\} \in \Pi_4$. In order to describe the complex $\mathcal{C}_x$ we consider the interval $(\perp, x)$ in Π_4, that is:

$$\begin{tikzcd}
 & \{1|234\} & \\
\{23|1|4\} \arrow[ur, dash] & \{24|1|3\} \arrow[u, dash] & \{34|1|2\} \arrow[ul, dash] \\
 & \perp \arrow[ul, dash] \arrow[u, dash] \arrow[ur, dash] &
\end{tikzcd}$$

We see that the chain complex $\mathcal{C}_{1|234}$ is of the form,

$$\mathcal{C}_{1|234}: \quad A \xleftarrow{\partial} B,$$

where A and B are of homological degree -1 and 0 respectively, and:

- A is one-dimensional with basis $\{\perp < 1|234\}$, and
- B is three-dimensional with basis,

$$\begin{aligned}
\{ \quad & a := \perp < 23|1|4 < 1|234, \\
& b := \perp < 24|1|3 < 1|234, \\
& c := \perp < 34|1|2 < 1|234 \quad \}.
\end{aligned}$$

It is easy to see that $\ker(\partial)$ is two-dimensional with basis $\{a - b, a - c\}$, and so the homology is,

$$H_0\left(\mathcal{C}_{1|234}\right) \cong \Bbbk^{\oplus 2}.$$

Notice that we recover the same calculations for any set partition x of type $(3, 1)$. That is,

$$H_0\left(\mathcal{C}_x\right) \cong \Bbbk^{\oplus 2},$$

for any $x \in \Pi_4$ with $\mathrm{type}(x) = (3,1)$. There are exactly four such elements, $1|234, 2|134, 3|124$ and $4|123$, therefore,

$$\mathcal{WH}_{(3,1)} \cong \bigoplus_{\substack{x \in \Pi_4 \\ \mathrm{type}(x)=(3,1)}} H_0(\mathcal{C}_x) \cong \Bbbk^{\oplus 8}.$$

This is an isomorphism on the level of vector spaces. However, the order complexes and their homology have a richer structure, which we now descirbe.

6.1.4 S_n-module structure on the Whitney homology

S_n-module structure on $H_{n-3}(\Pi_n)$.

Fix $n \in \mathbb{N}$. A set partition $x \in \Pi_n$ consists of a decomposition of $\{1, \ldots, n\}$ into disjoint blocks. The symmetric group S_n acts on Π_n by permuting the elements $\{1, \ldots, n\}$ of these decompositions. Furthermore, it is plain to see that this action is order-preserving, and therefore S_n acts on Π_n by poset automorphisms. Therefore, by Remark 6.1.5, the poset homology $H_*(\Pi_n)$ is an S_n-module.

It is worth spelling this action out in some detail. Consider a basis element c for the j-th chain group $\mathcal{C}_j(\Pi_n)$,

$$c = \perp < x_1 < \cdots < x_{j+1} < \top.$$

The S_n-action is defined as follows. For $\sigma \in \mathrm{S}_n$ define,

$$\sigma \cdot c := \perp < \sigma \cdot x_0 < \cdots < \sigma \cdot x_j < \top, \tag{6.4}$$

where the action $\sigma \cdot x_k$ is the action of S_n on Π_n. This S_n action is readily seen to the respect the differential on $\mathcal{C}(\Pi_n)$, and thus taking homology gives that $H_*(\Pi_n)$ inherits the structure of a S_n-module.

In addition, it is well-known that Π_n is Cohen-Macaulay of rank $n-1$, and as such, has homology concentrated in degree $n-3$. Stanley gave a description of its S_n-module structure.

Proposition 6.1.15 ([31], Theorem 7.3)**.** *For $n \in \mathbb{N}$, the poset homology $H_{n-3}(\Pi_n)$ is isomorphic as an S_n-module to $\epsilon_n \otimes \mathsf{Lie}_n$.*

Definition 6.1.16. We denote the S_n-module $H_{n-3}(\Pi_n)$ by π_n.

Example 6.1.17. Consider the case $n = 3$, where Π_3 has the following poset structure.

$$\begin{array}{ccc} & \top & \\ \{1|23\} & \{2|13\} & \{3|12\} \\ & \bot & \end{array}$$

with associated chain complex,

$$\mathcal{C}(\Pi_3): \quad \mathcal{C}_{-1}(\Pi_3) = \langle \bot < \top \rangle \xleftarrow{\partial} \mathcal{C}_0(\Pi_3) = \left\langle \begin{array}{c} a := \bot < 1|23 < \top, \\ b := \bot < 2|13 < \top, \\ c := \bot < 3|12 < \top \end{array} \right\rangle,$$

Similarly to Example 6.1.14 we see that,

$$V := \ker(\partial) = \langle a - b, a - c \rangle \cong \mathbb{k}^{\oplus 2}.$$

We can identify this as a representation of S_3 via its characters. In particular, it suffices to compute the action of $(12), (123) \in \mathrm{S}_3$ on the basis elements $a - b, a - c$ of V. Direct computation shows,

$$(12) \cdot a = b, (12) \cdot b = a, (12) \cdot c = c,$$

and,

$$(123) \cdot a = b, (123) \cdot b = c, (123) \cdot c = a,$$

whence we readily compute the characters,

$$\chi_V(12) = 0 \qquad \chi_V(123) = -1.$$

It follows (by inspecting the character table for S_3) that V can be identified with the irreducible S_3-module,

$$V \cong P_{(2,1)} = \ydiagram{2,1}$$

Induced module structure on $\mathcal{C}_\lambda$

We are motivated to enrich the structure of $\mathcal{C}_x$ for $x \in \Pi_n$. However, the symmetric group S_n does not act by automorphisms in analogy with Eq. 6.4. Indeed, given a chain,

$$\perp < x_1 < \cdots < x_{j+1} < x \in (\mathcal{C}_x)_j,$$

and an element $\sigma \in \mathrm{S}_n$, the j-chain,

$$\perp < \sigma \cdot x_1 < \cdots < \sigma \cdot x_{j+1} < \sigma \cdot x,$$

is not, in general, an element of $(\mathcal{C}_x)_j$. We require that $\sigma \cdot x = x$. To that end, we consider the stabilizer subgroup $\mathrm{Stab}_{\mathrm{S}_n}(x) \leq \mathrm{S}_n$.

First we make a simple observation about these stabilizer subgroups.

Lemma 6.1.18. *Let $n \in \mathbb{N}$ and $\lambda \vdash n$. For any two set partitions $x, x' \in \Pi_\lambda$ of the same type, there is an isomorphism,*

$$\mathrm{Stab}_{\mathrm{S}_n}(x) \cong \mathrm{Stab}_{\mathrm{S}_n}(x').$$

Proof. Let $x, x' \in \Pi_\lambda$ be two such set partitions. Note that there exists an element $\sigma \in \mathrm{S}_n$ such that $\sigma \cdot x = x'$. The isomorphism $\mathrm{Stab}_{\mathrm{S}_n}(x) \cong \mathrm{Stab}_{\mathrm{S}_n}(x')$ is given by conjugation by σ. $\square$

This lemma allows us to make the following definition.

Definition 6.1.19. Fix $n \in \mathbb{N}$ and $\lambda \vdash n$. Define G_λ to be the stabilizer subgroup,

$$G_\lambda := \mathrm{Stab}_{\mathrm{S}_n}(x),$$

where x is any element of type λ in Π_n.

It is useful to describe the stabilizers G_λ explicitly. Intuitively, the permutations of S_n that fix an element $x = (B_1, \dots, B_i)$ are those that

(i) Permute elements within the blocks B_k, and;

(ii) Permute blocks of the same size $B_k \leftrightarrow B_j$ where $|B_k| = |B_j|$.

In [32], Sundaram makes this intuition precise, giving a decomposition of G_λ in wreath products of the form $\mathrm{S}_i[\mathrm{S}_a]$.

Lemma 6.1.20 ([32], Theorem 1.4)**.** *Let $\lambda = 1^{m_1}2^{m_2} \cdots \vdash n$. The stabilizer subgroup G_λ factors as a product,*

$$G_\lambda \cong \prod_j \mathrm{S}_{m_j}[\mathrm{S}_j].$$

Remark 6.1.21.

1. We denote an element of $G_{(a^i)} \cong \mathrm{S}_i[\mathrm{S}_a]$ as $[\sigma; \underline{\tau}]$ where $\sigma \in \mathrm{S}_i$ and $\underline{\tau} = (\tau_1, \dots, \tau_{m_j}) \in \mathrm{S}_a^{\times i}$.

2. We describe the action of G_λ on Π_n. Consider an element $[\sigma; \underline{\tau}] \in \mathrm{S}_i[\mathrm{S}_a]$ and an element $x = (B_1| \cdots |B_i) \in \Pi_\lambda$ where $\lambda = (a^i)$. First there is a natural identification of the element $\tau_j \in \mathrm{S}_a$ with an element of $\mathrm{Sym}(B_j)$ for all $j = 1, \dots, i$. Now each block B_j of x is of the form,

$$B_j = \{b_{j1}, \dots, b_{jk}, \dots, b_{ja}\}.$$

We identify the permutation $\sigma \in \mathrm{S}_i$ with the element in $\mathrm{Sym}(B_1 \sqcup \cdots \sqcup B_i)$ sending $b_{jk} \mapsto b_{\sigma(j)k}$ for all $j = 1, \ldots, i$ and $k = 1, \ldots, a$. Under these identifications we have that $\sigma, \tau_1, \ldots, \tau_i \in \mathrm{Sym}(B_1 \sqcup \cdots \sqcup B_i)$ (which is naturally isomorphic to S_n) and as such we define

$$[\sigma; \underline{\tau}] \cdot x := \sigma \cdot \tau_1 \cdots \tau_i \cdot x.$$

Given two elements $[\sigma; \underline{\tau}], [\omega; \underline{\zeta}] \in \mathrm{S}_i[\mathrm{S}_a]$ we have,

$$[\sigma; \underline{\tau}] \cdot [\omega; \underline{\zeta}] = [\sigma\omega; \omega^{-1}(\underline{\tau})\underline{\zeta}],$$

where $\omega^{-1}(\underline{\tau}) := \left(\tau_{\omega^{-1}(1)}, \ldots, \tau_{\omega^{-1}(i)}\right)$. It is routine to verify that this defines an action of $\mathrm{S}_i[\mathrm{S}_a]$ on x. This action can be read off as a permutation of S_n. Indeed, there is a natural identification,

$$\mathrm{Sym}(B_1 \sqcup \cdots \sqcup B_j) \cong \mathrm{S}_n\,.$$

We use this identification to determine the action of $[\sigma; \underline{\tau}]$ on any element of Π_n.

Example 6.1.22. Fix $\lambda = (2, 2, 2) \vdash 6$ and $x = 12|34|56 \in \Pi_\lambda$. Consider,

$$\sigma = (123) \in \mathrm{S}_3,$$

and,

$$\underline{\tau} = (1, (12), (12)) \in \mathrm{S}_2^{\times 3}\,.$$

The element of S_6 corresponding to $[\sigma; \underline{\tau}]$ is determined as follows.

$$\sigma = (b_1 b_3 b_5)(b_2 b_4 b_6) \in \mathrm{Sym}(B_1 \sqcup B_2 \sqcup B_3),$$

and,

$$\left.\begin{array}{r} \tau_1 = (b_1)(b_2) \in \mathrm{Sym}(B_1) \\ \tau_2 = (b_3 b_4) \in \mathrm{Sym}(B_2) \\ \tau_3 = (b_5 b_6) \in \mathrm{Sym}(B_3) \end{array}\right\} \in \mathrm{Sym}(B_1 \sqcup B_2 \sqcup B_3).$$

Putting this together we get $[\sigma; \underline{\tau}] \cdot x$ by first applying $\underline{\tau}$ and then σ:

$$x = (12|34|56) \xrightarrow{\underline{\tau}} (12|43|65) \xrightarrow{\sigma} (43|65|12) =: [\sigma; \underline{\tau}] \cdot x,$$

and we see that the corresponding element of S_6 is,

$$[\sigma; \underline{\tau}] \leftrightarrow (145)(236).$$

3. Given an S_i-module V and a S_a-module W, the wreath product module $V[W]$ (see Eq. 2.3) admits the following action of $\mathrm{S}_i[\mathrm{S}_a]$. Let $[\sigma, \underline{\tau}] \in \mathrm{S}_i[\mathrm{S}_a]$ and

$$v \otimes (w_1 \otimes \cdots \otimes w_i) \in V[W].$$

Then,

$$[\sigma, \underline{\tau}] \cdot (v \otimes \big(w_1 \otimes \cdots \otimes w_i)) = \sigma \cdot v \otimes \big(\tau_1 w_{\sigma^{-1}(1)} \otimes \cdots \otimes \tau_i w_{\sigma^{-1}(i)}\big)$$

4. Recall that the wreath product $\mathrm{S}_i[\mathrm{S}_a]$ can be constructed as the normalizer in $\mathrm{S}_{i \cdot a}$ of $\mathrm{S}_a^{\times i}$. This gives us a natural identification of G_λ as a subgroup of S_n,

$$G_\lambda \cong \prod_j \mathrm{S}_{m_j}[\mathrm{S}_j] \hookrightarrow \prod_j \mathrm{S}_{m_j \cdot j} \hookrightarrow \mathrm{S}_n\,.$$

Lemma 6.1.23. *Let $\lambda \vdash n$ and $x \in \Pi_\lambda$. The complex $\mathcal{C}_x$ is a G_λ-module with the following action: let $\gamma \in G_\lambda \leq \mathrm{S}_n$ and,*

$$c = \perp < x_1 < \cdots < x_{j+1} < x,$$

a basis element for the j-th chain group $(\mathcal{C}_x)_j$. Then,

$$\gamma \cdot c := \perp < \gamma \cdot x_1 < \cdots < \gamma \cdot x_{j+1} < \gamma \cdot x = x.$$

where $\gamma \cdot x_k$ is determined by the action of S_n on Π_n.

Proof. This is routine and follows from the fact that the action of S_n on Π_n is by poset automorphisms. □

In [32] Sundaram gives an explicit description of the G_λ-module structure on the homology of the complex $\mathcal{C}_x$ in terms of the S_j-modules π_j.

Lemma 6.1.24 ([32], Theorem 1.7)**.** *Let x be a set partition of type $\lambda = 1^{m_1}2^{m_2}\cdots \vdash n$. The top homology of the interval $(\perp, x)$ is a G_λ-module isomorphic to,*

$$\bigotimes_j \mathcal{R}_{m_j}[\pi_j],$$

where,

$$\mathcal{R}(m_i) = \begin{cases} P_{(m_i)} & i \ \mathit{odd} \\ P_{(1^{m_i})} & i \ \mathit{even} \end{cases}.$$

Lemma 6.1.25. *Let $\lambda \vdash n$ and $x \in \Pi_\lambda$. There is a canonical isomorphism,*

$$\mathcal{C}_\lambda \cong \mathrm{Ind}_{G_\lambda}^{\mathrm{S}_n} \mathcal{C}_x.$$

Proof. We have that $\mathcal{C}_\lambda$ is a S_n-module that splits as a direct sum $\bigoplus_{x\in\Pi_\lambda} \mathcal{C}_x$ of subgroups that are permuted transitively by the S_n-action. Further we have that $\mathcal{C}_x$ is one of the summands, and G_λ is its stabilizer in S_n. This property is characteristic of induced modules by Lemma 2.1.15, and it follows that $\mathcal{C}_\lambda$ is canonically isomorphic to $\mathrm{Ind}_{G_\lambda}^{\mathrm{S}_n} \mathcal{C}_x$. □

Remark 6.1.26. It is worth describing the S_n-action on $\mathcal{C}_\lambda$ explicitly. Consider a basis element c for the j-th chain group $(\mathcal{C}_\lambda)_j$,

$$c = \perp < x_1 < \cdots < x_{j+1} < x,$$

where $x \in \Pi_\lambda$. Recall that the S_n-action is defined as follows. Let $\sigma \in \mathrm{S}_n$, then,

$$\sigma \cdot c = \perp < \sigma\cdot x_1 < \cdots < \sigma \cdot x_{j+1} < \sigma \cdot x,$$

where the action $\sigma \cdot x_k$ is the action of S_n on Π_n.

S_n-module structure on type and length selected Whitney homology.

The S_n-module structure on $H_{n-3}(\Pi_n)$ arose from the S_n-action on the poset Π_n. Notice that the action of S_n on Π_n transitively permutes set partitions of the same type. That is to say, for a given partition $\lambda \vdash n$, the type-selected subset Π_λ admits an action of S_n. This in turn implies that length-selected subset $\Pi_{i,n}$ admits an action of S_n. These action give rise to interesting S_n-modules $\mathcal{WH}_\lambda$ and $\mathcal{WH}_{i,n}$.

Lemma 6.1.27. *Fix $n \in \mathbb{N}$.*

1. *Let $\lambda \vdash n$. Then $\mathcal{WH}_\lambda$ is an S_n-module.*
2. *Let $i < n$. Then $\mathcal{WH}_{i,n}$ is an S_n-module.*

Proof. In Lemma 6.1.25 we showed that $\mathcal{C}_\lambda$ is an S_n-module. Furthermore, the action of S_n respects the differential, and thus the homology $\mathcal{WH}_\lambda$ inherits the structure of an S_n-module (see Remark 6.1.26). The complex $\mathcal{C}_{i,n}$ is a sum of complexes of the form $\mathcal{C}_\lambda$, and the action of S_n is diagonal. The second statement follows. □

Example: S_4**-module structure of** $\mathcal{WH}_{2,4}$

We give an explicit description of the S_4-module structure of the 2nd Whitney homology of Π_4,

$$\mathcal{WH}_{2,4} \cong \mathcal{WH}_{(3,1)} \oplus \mathcal{WH}_{(2,2)}.$$

We separately compute the type-selected Whitney homologies $\mathcal{WH}_{(3,1)}$ and $\mathcal{WH}_{(2,1,1)}$.

Type $\lambda = (3,1)$. We compute the S_4-module structure of $\mathcal{WH}_{(3,1)}$ following from Example 6.1.14. We recall (and extend) the notation introduced there by naming the 0-chains in $\mathcal{C}_{(3,1)}$.

$(\mathcal{C}_{1\|234})_0$	$(\mathcal{C}_{2\|134})_0$
$a := \perp < 1\|4\|23 < 1\|234$	$d := \perp < 2\|4\|13 < 2\|134$
$b := \perp < 1\|3\|24 < 1\|234$	$e := \perp < 2\|3\|14 < 2\|134$
$c := \perp < 1\|4\|23 < 1\|234$	$f := \perp < 2\|1\|34 < 2\|134$

$(\mathcal{C}_{3\|124})_0$	$(\mathcal{C}_{4\|123})_0$
$g := \perp < 3\|4\|12 < 3\|124$	$j := \perp < 4\|3\|12 < 4\|123$
$h := \perp < 3\|2\|14 < 3\|124$	$k := \perp < 4\|2\|13 < 4\|123$
$i := \perp < 3\|1\|24 < 3\|124$	$l := \perp < 4\|1\|23 < 4\|123$

In Example 6.1.14 we showed that $\mathcal{WH}_{(3,1)}$ is 8-dimensional with basis,

$$\mathcal{WH}_{(3,1)} = \langle a-b, a-c, d-e, d-f, g-h, g-i, j-k, j-l \rangle.$$

We will use some elementary character theory to determine the decomposition of $\mathcal{WH}_{(3,1)}$ into irreducible S_4-modules (see [19], for the relevant background on character theory). It suffices to consider the action of $e, (12), (12)(34), (123), (1234) \in \mathrm{S}_4$ on a basis for $\mathcal{WH}_{(3,1)}$.

As a first step we record the relevant actions on the 0-chains in the following multiplication table.

	a	b	c	d	e	f	g	h	i	j	k	i
(12)	d	e	f	a	b	c	g	i	h	j	l	k
$(12)(34)$	e	d	f	b	a	c	j	l	k	g	i	h
(123)	d	f	e	g	i	h	a	b	c	l	j	k
(1234)	f	d	e	i	g	h	l	j	k	a	b	c

(6.5)

A straightforward (if not somewhat verbose) calculation now reveals the character of the representation $\mathcal{WH}_{(3,1)}$ which we denote χ_W. For instance, we see that $(12) \in \mathrm{S}_4$ acts on the basis element $(a - b)$ as,

$$(12) \cdot (a - b) = (d - e).$$

Continuing in this way we compute the characters of each conjugacy class of S_4, recording them together with the character table for S_4 (Example 2.1.21).

	1	(12)	(12)(34)	(123)	(1234)
$\chi_{(4)}$	1	1	1	1	1
$\chi_{(1,1,1,1)}$	1	−1	1	1	−1
$\chi_{(3,1)}$	3	1	−1	0	−1
$\chi_{(2,1,1)}$	3	−1	−1	0	1
$\chi_{(2,2)}$	2	0	2	−1	0
χ_W	8	0	0	−1	0

Taking the inner product of χ_W with $\chi_{(3,1)}, \chi_{(2,1,1)}$ and $\chi_{(2,2)}$,

$$\begin{aligned}
\langle \chi_W, \chi_{(3,1)} \rangle &= \frac{1}{24}(1 \cdot 8 \cdot 3 + 6 \cdot 0 \cdot 1 + 3 \cdot 0 \cdot -1 + 8 \cdot -1 \cdot 0 + 6 \cdot 0 \cdot -1) = 1, \\
\langle \chi_W, \chi_{(2,1,1)} \rangle &= \frac{1}{24}(1 \cdot 8 \cdot 3 + 6 \cdot 0 \cdot -1 + 3 \cdot 0 \cdot -1 + 8 \cdot -1 \cdot 0 + 6 \cdot 0 \cdot 1) = 1, \\
\langle \chi_W, \chi_{(2,2)} \rangle &= \frac{1}{24}(1 \cdot 8 \cdot 2 + 6 \cdot 0 \cdot 0 + 3 \cdot 0 \cdot 2 + 8 \cdot -1 \cdot -1 + 6 \cdot 0 \cdot 0) = 1,
\end{aligned} \tag{6.6}$$

gives the following decomposition of $\mathcal{WH}_{(3,1)}$.

$$\begin{aligned}
\mathcal{WH}_{(3,1)} &\cong P_{(3,1)} \oplus P_{(2,1,1)} \oplus P_{(2,2)} \\
&= \ydiagram{3,1} \oplus \ydiagram{2,1,1} \oplus \ydiagram{2,2}\,.
\end{aligned}$$

Type $\lambda = (2,2)$. We compute the S_4-module structure of $\mathcal{WH}_{(2,2)}$. The 0-chains in $\mathcal{C}_{(2,2)}$ are.

$(\mathcal{C}_{12\|34})_0$	$(\mathcal{C}_{13\|24})_0$	$(\mathcal{C}_{14\|23})_0$
$a := \perp < 1\|2\|34 < 12\|34$	$c := \perp < 1\|3\|24 < 13\|24$	$e := \perp < 1\|4\|23 < 14\|23$
$b := \perp < 12\|3\|4 < 12\|34$	$d := \perp < 13\|2\|4 < 13\|24$	$f := \perp < 14\|2\|3 < 14\|23$

Similarly to Example 6.1.14 we have that $\mathcal{WH}_{(2,2)}$ is 3-dimensional with basis,

$$\mathcal{WH}_{(2,2)} = \langle a - b, c - d, e - f \rangle. \tag{6.7}$$

As above we record the relevant actions on the 0-chains in a multiplication table.

	a	b	c	d	e	f
(12)	a	b	f	e	d	c
(12)(34)	a	b	d	c	f	e
(123)	f	e	a	b	d	c
(1234)	f	e	d	c	a	b

With this it is easy to compute the action on basis elements of $\mathcal{WH}_{(2,2)}$. For example,

$$(12) \cdot (a - b) = (a - b)$$
$$(12) \cdot (c - d) = (f - e) = -(e - f)$$
$$(12) \cdot (e - f) = (d - c) = -(c - d)$$

Continuing in this fashion we are able to explicitly describe the matrices corresponding to the representation determined by $\mathcal{WH}_{(2,2)}$ in terms of the basis given in Eq. 6.7.

$$
\begin{gathered}
(12) \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \qquad (12)(34) \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
(123) \leftrightarrow \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \qquad (1234) \leftrightarrow \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}
\end{gathered}
\tag{6.8}
$$

We obtain the character for $\mathcal{WH}_{(2,2)}$, which we denote χ_U as,

	1	(12)	(12)(34)	(123)	(1234)
χ_U	3	1	-1	0	-1

which we recognize as the character for $P_{(3,1)}$. That is, we have the isomorphism,

$$\mathcal{WH}_{(2,2)} \cong P_{(3,1)} = \ydiagram{3,1}$$

The following result of Lehrer-Solomon describes the representations of $\mathcal{WH}_\lambda$ in terms of the twisted Lie operad.

Proposition 6.1.28 ([21], Theorem 4.5)**.** *Fix $n \in \mathbb{N}$ and write $\lambda \vdash n$ in terms of its exponents $\lambda = 1^{m_1} 2^{m_2} \cdots$. There is an isomorphism of S_n-modules,*

$$\mathcal{WH}_\lambda \cong \mathop{\circledast}_i \left(\mathcal{R}(m_i) \circ \widehat{\mathsf{Lie}}_i \right), \tag{6.9}$$

where,

$$\mathcal{R}(m_i) = \begin{cases} P_{(m_i)} & i \text{ odd} \\ P_{(1^{m_i})} & i \text{ even} \end{cases}.$$

Example 6.1.4 (Continued). Write $\lambda = (3,1) = 1^1 3^1$ giving $m_1 = m_3 = 1$ and $\mathcal{R}(m_1) = \mathcal{R}(m_3) = \square$. Plugging into Eq. 6.9 we get that,

$$\mathcal{WH}_{(3,1)} \cong \left(\ydiagram{1} \circ \ydiagram{1} \right) \circledast \left(\ydiagram{1} \circ \ydiagram{2,1} \right) = \ydiagram{3,1} \oplus \ydiagram{2,1,1} \oplus \ydiagram{2,2}.$$

Similarly, writing $\lambda = (2,2) = 2^2$ gives,

$$\mathcal{WH}_{(2,2)} \cong \left(\ydiagram{1,1} \circ \ydiagram{2} \right) = \ydiagram{3,1},$$

as expected.

6.2 Extending the action

Our goal in this section is to introduce an action on the blocks of a set partition. We therefore need to keep track of the order of the blocks.

6.2.1 Ordered set partitions

Definition 6.2.1. Fix $n \in \mathbb{N}$. An ordered set partition $\widetilde{x} = (B_1, \ldots, B_i)$ of $\{1, \ldots, n\}$ is a decomposition,

$$\{1, \ldots, n\} = \bigsqcup_{j=1}^{i} B_j,$$

into disjoint sets B_j which we also refer to as blocks, together with an order on the blocks.

Let $\widetilde{\Pi}_n$ denote the set of all ordered set partitions of $\{1, \ldots, n\}$. We denote an ordered set partition with $\|$ bars. For example,

$$(12 \parallel 34) \neq (34 \parallel 12) \in \widetilde{\Pi}_4$$

Definition 6.2.2. Define the support map by,

$$\begin{aligned} \mathrm{supp}: \quad \widetilde{\Pi}_n \quad &\to \quad \Pi_n \\ (B_1, \ldots, B_l) &\mapsto \{B_1, \ldots, B_l\} \end{aligned}$$

Then for $x \in \Pi_n$, the preimage $\mathrm{supp}^{-1}(x) \subseteq \widetilde{\Pi}_n$ consists of all ordered set partitions of support x. Define,

$$\widetilde{\Pi}_\lambda = \bigcup_{\substack{x \in \Pi_n \\ \mathrm{type}(x) = \lambda}} \mathrm{supp}^{-1}(x).$$

Let $\widetilde{x} \in \widetilde{\Pi}_n$ an ordered set partition. Its type is defined to agree with the type of its support, namely,

$$\mathrm{type}(\widetilde{x}) := \mathrm{type}(\mathrm{supp}(\widetilde{x})).$$

Example 6.2.3. As a simple example to familiarize ourselves with the notation, consider the ordered set partitions $(12 \parallel 34)$ and $(34 \parallel 12) \in \widetilde{\Pi}_4$. We have,

$$\begin{tikzcd} 12 \parallel 34 \arrow[dr, mapsto, "\mathrm{supp}"] & \\ & 12|34 \\ 34 \parallel 12 \arrow[ur, mapsto, "\mathrm{supp}"'] & \end{tikzcd}$$

and $\mathrm{supp}^{-1}(\{12|34\}) = \{(12 \parallel 34), (34 \parallel 12)\} \subseteq \widetilde{\Pi}_4$. Let $\lambda = (2, 2)$ be the type of $12|34 \in \Pi_4$. Then,

$$\begin{aligned} \widetilde{\Pi}_{(2,2)} = \{&(12 \parallel 34), (34 \parallel 12), \\ &(13 \parallel 24), (24 \parallel 13), \\ &(14 \parallel 23), (14 \parallel 23)\} \subseteq \widetilde{\Pi}_4. \end{aligned}$$

6.2.2 The extended action

We are now ready to describe an extension of the S_n-module structure on Π_n.

Lemma 6.2.4. *Fix a partition $\lambda \vdash n$ of length i. The set $\widetilde{\Pi}_\lambda \subseteq \widetilde{\Pi}_n$ admits an action of $\mathrm{S}_i \times \mathrm{S}_n$ with:*

1. *Left action of S_i on blocks. Concretely, let $\widetilde{x} = (B_1, \ldots, B_i) \in \widetilde{\Pi}_\lambda$ and $\tau \in \mathrm{S}_i$, then,*
$$\tau \cdot (B_1, \ldots, B_i) = (B_{\tau \cdot 1}, \ldots, B_{\tau \cdot i}). \tag{6.10}$$

2. *Right action of S_n on elements. Concretely, let $\widetilde{x} = (B_1, \ldots, B_i) \in \widetilde{\Pi}_\lambda$ and $\sigma \in \mathrm{S}_n$, then,*
$$(B_1, \ldots, B_i) \cdot \sigma = (B_1 \cdot \sigma, \ldots, B_i \cdot \sigma), \tag{6.11}$$
where $B_j \cdot \sigma = \{b_1, \ldots, b_k\} \cdot \sigma = \{\sigma^{-1} \cdot b_1, \ldots, \sigma^{-1} \cdot b_k\}$.

Proof. It is routine to show that Eqs. 6.10 and 6.11 define left and right actions of S_i and S_n on $\widetilde{\Pi}_\lambda$ (resp.) noting that permuting blocks and permuting elements within a block of an ordered set partition does not alter its type. It remains to show that these actions commute. Let $\tau \in \mathrm{S}_i, \sigma \in \mathrm{S}_n$ and $(B_1, \ldots, B_i) \in \widetilde{\Pi}_\lambda$. Then,

$$\begin{aligned}
\tau \cdot ((B_1, \ldots, B_i) \cdot \sigma) &= \tau \cdot (B_1 \cdot \sigma, \ldots, B_i \cdot \sigma) \\
&= (B_{\tau \cdot 1} \cdot \sigma, \ldots, B_{\tau \cdot i} \cdot \sigma) \\
&= (B_{\tau \cdot 1}, \ldots, B_{\tau \cdot i}) \cdot \sigma \\
&= (\tau \cdot (B_1, \ldots, B_i)) \cdot \sigma,
\end{aligned}$$

as required. $\square$

Example 6.2.5. Let $\lambda = (2, 2) \vdash 4$ and consider the set $\widetilde{\Pi}_\lambda \subseteq \widetilde{\Pi}_4$ from Example 6.2.3. Consider the ordered set partition $(12 \parallel 34) \in \widetilde{\Pi}_\lambda$.

- (left-action) Let $(12) \in \mathrm{S}_i = \mathrm{S}_2$. Then,
$$(12) \cdot (12 \parallel 34) = (34 \parallel 12).$$

- (right-action) Let $(13) \in \mathrm{S}_n = \mathrm{S}_4$. Then,

$$(12 \parallel 34) \cdot (13) = (23 \parallel 14).$$

This structure on $\widetilde{\Pi}_\lambda \subset \widetilde{\Pi}_n$ motivates us to enlarge the type complexes $\mathcal{C}_\lambda$ and $\mathcal{C}_{i,n}$. We spell this out in the next section.

6.2.3 Extended Whitney complexes and Whitney homology.

There is a map from the lattice of set partitions to the category of chain complexes,

$$\mathcal{C} : \Pi_n \to \mathsf{dgVect},$$

taking x to the complex $\mathcal{C}_x$. Extend this to a map,

$$\widetilde{\mathcal{C}} : \widetilde{\Pi}_n \to \mathsf{dgVect},$$

defined as the composition $\widetilde{\mathcal{C}} = \mathcal{C} \circ \operatorname{supp}$. In other words, we assign to each ordered set partition $\widetilde{x} \in \widetilde{\Pi}_n$ the chain complex associated to the underlying unordered set partition $\operatorname{supp}(\widetilde{x}) \in \Pi_n$. Denote the complex $\widetilde{\mathcal{C}}(\widetilde{x})$ simply as $\mathcal{C}_{\widetilde{x}}$.

Similarly, we define the extended counterparts to the type-complex $\mathcal{C}_\lambda$ and the rank-selected order complex $\mathcal{C}_{i,n}$ in Definition 6.1.12.

Definition 6.2.6. Let $\lambda \vdash n$ of length i.

1. Define the extended type-selected order complex $\widetilde{\mathcal{C}}_\lambda$ as,

$$\widetilde{\mathcal{C}}_\lambda = \bigoplus_{\substack{\widetilde{x} \in \widetilde{\Pi}_n \\ \operatorname{type}(\widetilde{x}) = \lambda}} \mathcal{C}_{\widetilde{x}} = \bigoplus_{\widetilde{x} \in \widetilde{\Pi}_\lambda} \mathcal{C}_{\widetilde{x}}.$$

   The extended Whitney homology of type λ, $\widetilde{\mathcal{WH}}_\lambda$, is the homology the complex $\widetilde{\mathcal{C}}_\lambda$.

2. We again group these complexes by length, defining the extended length-selected order complex $\widetilde{\mathcal{C}}_{i,n}$ as,

$$\widetilde{\mathcal{C}}_{i,n} := \bigoplus_{\substack{\lambda \vdash n \\ l(\lambda)=i}} \widetilde{\mathcal{C}}_\lambda,$$

The i-th extended Whitney homology, $\widetilde{\mathcal{WH}}_{i,n}$, is the homology of the complex $\widetilde{\mathcal{C}}_{i,n}$.

Remark 6.2.7. We note that the chain group $(\widetilde{\mathcal{C}}_\lambda)_j$ has basis elements consisting of chains in $\mathcal{C}_{\widetilde{x}} = \mathcal{C}_x$ labelled by an ordering of the blocks of $x \in \Pi_n$. We denote such a basis element,

$$\perp < x_1 < \cdots < x_j < \widetilde{x},$$

where $\widetilde{x} \in \widetilde{\Pi}_n$. This corresponds to the element $(\perp < x_1 < \cdots < x_j < x)$ in the summand labelled by $\widetilde{x}$.

Example 6.2.8. The complex $\widetilde{\mathcal{C}}_{(2,2)}$ admits the following decomposition,

$$\widetilde{\mathcal{C}}_{(2,2)} = \widetilde{\mathcal{C}}_{12\|34} \oplus \widetilde{\mathcal{C}}_{34\|12} \oplus \widetilde{\mathcal{C}}_{13\|24} \oplus \widetilde{\mathcal{C}}_{24\|13} \oplus \widetilde{\mathcal{C}}_{14\|23} \oplus \widetilde{\mathcal{C}}_{23\|14}. \tag{6.12}$$

Consider the 0-chain,

$$\perp < 1|2|34 < 12|34,$$

a basis element in $(\mathcal{C}_{12|34})_0$, This has two corresponding extended counterparts,

$$\begin{aligned} \perp < 1|2|34 < 12 \parallel 34 \quad &\leftrightarrow \quad (\perp < 1|2|34 < 12|34) \in \widetilde{\mathcal{C}}_{12\|34}, \\ \perp < 1|2|34 < 34 \parallel 12 \quad &\leftrightarrow \quad (\perp < 1|2|34 < 12|34) \in \widetilde{\mathcal{C}}_{34\|12}, \end{aligned}$$

in $\widetilde{\mathcal{C}}_{(2,2)}$ belonging to the first two summands in Eq. 6.12.

$(\mathrm{S}_i, \mathrm{S}_n)$-bimodule structure on $\widetilde{\mathcal{C}}_\lambda$ and $\widetilde{\mathcal{C}_{i,n}}$. We describe the $(\mathrm{S}_i, \mathrm{S}_n)$-bimodule structure on the complex $\widetilde{\mathcal{C}}_\lambda$. It suffices to specify the action on a basis for the j-th chain group $(\widetilde{\mathcal{C}}_\lambda)_j$. Let,

$$c = (\perp < x_1 < \cdots < x_{j+1} < \widetilde{x}) \in (\widetilde{\mathcal{C}}_\lambda)_j,$$

denote such a basis element, where $\widetilde{x} \in \widetilde{\Pi}_n$ is of type λ.

1. S_i acts on $\widetilde{\mathcal{C}}_\lambda$ by permuting the summands in,

$$\bigoplus_{\widetilde{x} \in \widetilde{\Pi}_\lambda} \mathcal{C}_{\widetilde{x}}.$$

   Concretely, let $\tau \in \mathrm{S}_i$. Then,

$$\tau \cdot c := (\perp < x_1 < \cdots < x_{j+1} < \tau \cdot \widetilde{x}) \tag{6.13}$$

2. S_n acts simultaneously by permuting summands as above, and also by the unordered action on the chain. Concretely, let $\sigma \in \mathrm{S}_n$. Then,

$$c \cdot \sigma := (\perp < x_1 \cdot \sigma < \cdots < x_{j+1} \cdot \sigma < \widetilde{x} \cdot \sigma), \tag{6.14}$$

   where $x_k \cdot \sigma := \sigma^{-1} \cdot x_k$ is the usual action of S_n on Π_n (written as a right action).

Proposition 6.2.9. *Fix a partition $\lambda \vdash n$ of length i.*

1. *The Whitney homology $\widetilde{\mathcal{WH}}_\lambda$ admits the structure of an $(\mathrm{S}_i, \mathrm{S}_n)$-bimodule.*
2. *The Whitney homology $\widetilde{\mathcal{WH}}_{i,n}$ admits the structure of an $(\mathrm{S}_i, \mathrm{S}_n)$-bimodule.*

Proof. By the same argument as Lemma 6.1.27, the action of S_n respects the differential on $\widetilde{\mathcal{C}}_\lambda$. The S_i action only permutes summands, and is readily seen to commute with the differential. We therefore have that the homology of these complexes, $\widetilde{\mathcal{WH}}_\lambda$, inherit the structure of a $(\mathrm{S}_i, \mathrm{S}_n)$-bimodule, as desired. Finally, $\mathrm{S}_i \times \mathrm{S}_n$ acts diagonally on,

$$\widetilde{\mathcal{C}}_{i,n} := \bigoplus_{\substack{\lambda \vdash n \\ l(\lambda)=i}} \widetilde{\mathcal{C}}_\lambda.$$

The differential $\partial_{i,n}$ on $\widetilde{\mathcal{C}}_{i,n}$ splits over this sum as $\partial_{i,n} = \oplus \partial_\lambda$, where ∂_λ is the differential on the extended type complex $\widetilde{\mathcal{C}}_\lambda$. The result follows upon taking homology, as above. $\square$

Example: $(\mathrm{S}_2, \mathrm{S}_4)$**-bimodule structure of** $\widetilde{\mathcal{WH}}_{2,4}$

In Example 6.1.4 we computed the S_4-module structure of $\mathcal{WH}_{2,4}$. Now we consider its extended counterpart $\widetilde{\mathcal{WH}}_{2,4}$ which is an $(\mathrm{S}_2, \mathrm{S}_4)$-bimodule satisfying,

$$\widetilde{\mathcal{WH}}_{2,4} \cong \widetilde{\mathcal{WH}}_{(2,2)} \oplus \widetilde{\mathcal{WH}}_{(3,1)}.$$

Type $(2,2)$**.** The complex $\widetilde{\mathcal{C}}_{(2,2)}$ is the sum of complexes,

$$\widetilde{\mathcal{C}}_{(2,2)} = \widetilde{\mathcal{C}}_{(12\|34)} \oplus \widetilde{\mathcal{C}}_{(34\|12)} \oplus \widetilde{\mathcal{C}}_{(13\|24)} \oplus \widetilde{\mathcal{C}}_{(24\|13)} \oplus \widetilde{\mathcal{C}}_{(14\|23)} \oplus \widetilde{\mathcal{C}}_{(23\|14)},$$

where for each $\widetilde{x} \in \widetilde{\Pi}_{(2,2)}$ the complex $\widetilde{\mathcal{C}}_{\widetilde{x}}$ is of the form,

$$\mathcal{C}_{\widetilde{x}} : \quad (\mathcal{C}_{\widetilde{x}})_{-1} \xleftarrow{\partial} (\mathcal{C}_{\widetilde{x}})_0 .$$

For each $\widetilde{x} \in \widetilde{\Pi}_\lambda$ the -1-chain group $(\mathcal{C}_{\widetilde{x}})_{-1}$ is one-dimensional with basis $\perp < \widetilde{x}$. We introduce some notation for the 0-chains.

$(\mathcal{C}_{12\|34})_0$	$(\mathcal{C}_{13\|24})_0$	$(\mathcal{C}_{14\|23})_0$
$a := \perp < 1\|2\|34 < 12 \parallel 34$	$c := \perp < 1\|3\|24 < 13 \parallel 24$	$e := \perp < 1\|4\|23 < 14 \parallel 23$
$b := \perp < 12\|3\|4 < 12 \parallel 34$	$d := \perp < 13\|2\|4 < 13 \parallel 24$	$f := \perp < 14\|2\|3 < 14 \parallel 23$

$(\mathcal{C}_{34\|12})_0$	$(\mathcal{C}_{24\|13})_0$	$(\mathcal{C}_{23\|14})_0$
$a' := \perp < 1\|2\|34 < 34 \parallel 12$	$c' := \perp < 1\|3\|24 < 24 \parallel 13$	$e' := \perp < 1\|4\|23 < 23 \parallel 14$
$b' := \perp < 12\|3\|4 < 34 \parallel 12$	$d' := \perp < 13\|2\|4 < 24 \parallel 13$	$f' := \perp < 14\|2\|3 < 23 \parallel 14$

As in Example 6.1.4 we see,

$$\widetilde{\mathcal{WH}}_{(2,2)} = H_0\left(\widetilde{\mathcal{C}}_{(2,2)}\right) \cong \bigoplus_{\widetilde{x} \in \widetilde{\Pi}_{(2,2)}} H_0(\mathcal{C}_{\widetilde{x}}) \cong \langle a-b, a'-b', c-d, c'-d', e-f, e'-f' \rangle.$$

We see that $\widetilde{\mathcal{WH}}_{(2,2)}$ is a 6-dimensional vector space. Moreover, by Proposition 6.2.9, this vector space is an $(\mathrm{S}_2, \mathrm{S}_4)$-bimodule. We compute the S_4-module structure and S_2-module structure separately.

As an S_4-module. Recall from Eq. 6.14 the (right) action of $\sigma \in \mathrm{S}_n$ on j-chains is given by,

$$(\perp < x_1 < \ldots < x_{j-1} < \widetilde{x}) \cdot \sigma = (\perp < x_1 \cdot \sigma < \ldots < x_{j-1} \cdot \sigma < \widetilde{x} \cdot \sigma).$$

where $x_k \cdot \sigma := \sigma^{-1} \cdot x_k$ is the usual action of S_n on Π_n (written as a right action).

For example, consider the right action of $(12) \in \mathrm{S}_4$ on the 0-chain $a \in (\mathcal{C}_{12\|34})_0$:

$$a \cdot (12) = (\perp < 1|2|34 < 12 \parallel 34) \cdot (12) = (\perp < 2|1|34 < 21 \parallel 34) = a$$

and on the 0-chain $c \in (\mathcal{C}_{13\|24})_0$:

$$c \cdot (12) = (\perp < 1|3|24 < 13 \parallel 24) \cdot (12) = (\perp < 2|3|14 < 23 \parallel 14) = f'$$

Continuing in this fashion we are able to record the action of each conjugacy class in S_4 on the chains $a, a', b, \cdots, f, f'$ as follows.

	a	a'	b	b'	c	c'	d	d'	e	e'	f	f'
(12)	a	a'	b	b'	f'	f	e'	e	d'	d	c'	c
(12)(34)	a	a'	b	b'	d'	d	c'	c	f'	f	e'	e
(132)	f'	f	e'	e	a	a'	b	b'	d'	d	c'	c
(1432)	f'	f	e'	e	d'	d	c'	c	a	a'	b	b'

This action on the chains induces an action on the homology. For example, consider the basis elements $a - b$ and $c - d$. We compute:

$$(a - b) \cdot (12) = (a \cdot (12) - b \cdot (12)) = a - b$$

$$(c - d) \cdot (12) = (c \cdot (12) - d \cdot (12)) = f' - e' = -(e' - f')$$

Fixing an order on our basis elements,

$$(a - b, a' - b', c - d, c' - d', e - f, e' - f'),$$

we can give matrix representations of our conjugacy classes as follows.

$$(12) \leftrightarrow \left(\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{array}\right) \qquad (12)(34) \leftrightarrow \left(\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right)$$

$$(132) \leftrightarrow \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{array}\right) \qquad (1432) \leftrightarrow \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right)$$

Note. It is interesting to compare these matrices with the analogues in Example 6.1.4. For example, consider the element $(12) \in \mathrm{S}_4$. Let $\rho : \mathrm{S}_4 \to \mathrm{End}(\mathcal{WH}_{(2,2)})$ denote the usual Whitney homology representation, and $\widetilde{\rho} : \mathrm{S}_4 \to \mathrm{End}(\widetilde{\mathcal{WH}}_{(2,2)})$ its extended counterpart. Then,

$$\rho(12) = \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 0 & -1 \\ \hline 0 & -1 & 0 \end{array}\right) \quad \rightsquigarrow \quad \widetilde{\rho}(12) = \left(\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{array}\right)$$

A 1 in the matrix $\rho(12)$ is either mapped to,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

or,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

depending on whether the ordered chain $(ab \parallel cd)$ is reversed or preserved under the action of $\sigma \in \mathrm{S}_4$. Similarly for -1's.

We can now compute the character $\chi_{\widetilde{U}}$ of $\widetilde{\mathcal{WH}}_{(2,2)}$ as an S_4-module. For convenience, we record the full character table for S_4.

	1	(12)	(12)(34)	(123)	(1234)
$\chi_{(4)}$	1	1	1	1	1
$\chi_{(1,1,1,1)}$	1	-1	1	1	-1
$\chi_{(3,1)}$	3	1	-1	0	-1
$\chi_{(2,1,1)}$	3	-1	-1	0	1
$\chi_{(2,2)}$	2	0	2	-1	0
$\chi_{\widetilde{U}}$	6	2	2	0	0

As in Example 6.1.4, we use the inner product on the space of characters to determine the irreducible S_4-modules appearing in $\widetilde{\mathcal{WH}}_{(2,2)}$.

$$\begin{aligned}
\langle \chi_{\widetilde{U}}, \chi_{(4)} \rangle &= \frac{1}{24}(1 \cdot 6 \cdot 1 + 6 \cdot 2 \cdot 1 + 3 \cdot 2 \cdot 1) = 1, \\
\langle \chi_{\widetilde{U}}, \chi_{(3,1)} \rangle &= \frac{1}{24}(1 \cdot 6 \cdot 3 + 6 \cdot 2 \cdot 1 + 3 \cdot 2 \cdot (-1)) = 1, \\
\langle \chi_{\widetilde{U}}, \chi_{(2,2)} \rangle &= \frac{1}{24}(1 \cdot 6 \cdot 2 + 6 \cdot 2 \cdot 0 + 3 \cdot 2 \cdot 2) = 1.
\end{aligned}$$

Thus we have the isomorphism of S_4-modules,

$$\widetilde{\mathcal{WH}}_{(2,2)} \cong \ydiagram{4} \oplus \ydiagram{3,1} \oplus \ydiagram{2,2}$$

As an S_2-module. Recall from Eq. 6.13 the (left) action of $\tau \in S_i$ on j-chains is given by,

$$\tau \cdot (\perp < x_1 < \ldots < x_{j-1} < \widetilde{x}) = (\perp < x_1 < \ldots < x_{j-1} < \tau \cdot \widetilde{x}),$$

where $\tau \cdot \widetilde{x}$ acts by permuting the blocks of $\widetilde{x}$. We see that $(12) \in S_2$ acts on the 0-chains as follows.

$$(12) : a \leftrightarrow a' \quad b \leftrightarrow b' \quad c \leftrightarrow c' \quad d \leftrightarrow d' \quad e \leftrightarrow e' \quad f \leftrightarrow f',$$

and we can immediate write the matrix representation of (12) as,

$$(12) \leftrightarrow \left(\begin{array}{cc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

Since this is a traceless matrix, and the only irreducible representations of S_2 are $P_{(2)}$ and $P_{(1,1)}$ on which (12) has trace 1 and -1 (resp.), we see immediately that as an S_2-module $\widetilde{\mathcal{WH}}_{(2,2)}$ admits the following decomposition into irreducibles.

$$\widetilde{\mathcal{WH}}_{(2,2)} \cong \left(\ydiagram{2} \oplus \ydiagram{1,1} \right)^{\oplus 3}$$

As an (S_2, S_4)-bimodule. We have computed separately the irreducible decompositions of $\widetilde{\mathcal{WH}}_{(2,2)}$ into S_2-modules and S_4-modules. These structures are compatible and $\widetilde{\mathcal{WH}}_{(2,2)}$ admits a decomposition into irreducible $\mathrm{S}_2 \times \mathrm{S}_4$-modules, which are of the form,

$$P_\mu \otimes P_\lambda,$$

where $\mu \vdash 2, \lambda \vdash 4$. For dimension reasons we see there are only two choices for such a decomposition. Namely, either,

$$\widetilde{\mathcal{WH}}_{(2,2)} \cong \ydiagram{1,1} \otimes \left(\ydiagram{4} \oplus \ydiagram{2,2} \right) \bigoplus \ydiagram{2} \otimes \ydiagram{3,1}$$

or,

$$\widetilde{\mathcal{WH}}_{(2,2)} \cong \ydiagram{2} \otimes \left(\ydiagram{4} \oplus \ydiagram{2,2} \right) \bigoplus \ydiagram{1,1} \otimes \ydiagram{3,1}$$

To determine which is correct we compute the S_2-invariants of $\widetilde{\mathcal{WH}}_{(2,2)}$ which is readily seen to have basis,

$$\left(\widetilde{\mathcal{WH}}_{(2,2)}\right)^{\mathrm{S}_2} = \langle (a-b) + (a'-b'), (c-d) + (c'-d'), (e-f) + (e'-f') \rangle.$$

This is a 3-dimensional subrepresentation of the S_4-module $\widetilde{\mathcal{WH}}_{(2,2)}$. Recycling the computations above we get the following matrix representations,

$$(12) \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \qquad (12)(34) \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$(132) \leftrightarrow \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \qquad (1432) \leftrightarrow \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

giving the character $\chi_{\mathrm{Res}\,\widetilde{U}}$,

	1	(12)	(12)(34)	(123)	(1234)
$\chi_{\mathrm{Res}\,\widetilde{U}}$	3	1	-1	0	-1

which we recognize as the character for $P_{(3,1)}$. Putting this together, we have computed the decomposition of $\widetilde{\mathcal{WH}}_{(2,2)}$ into irreducible $(\mathrm{S}_2, \mathrm{S}_4)$-bimodules as,

$$\widetilde{\mathcal{WH}}_{(2,2)} \cong \ydiagram{1,1} \otimes \left(\ydiagram{4} \oplus \ydiagram{2,2} \right) \quad \bigoplus \quad \ydiagram{2} \otimes \ydiagram{3,1}\,.$$

Type $(3,1)$**.** We make a similar calculation for the extended Whitney homology of type $(3,1)$. However, for the sake of brevity, we don't record quite as much detail as in the previous case. In particular, it is not necessary to compute the matrix embeddings as we did above, only the fixed points which correspond to the trace.

As an S_4**-module.** We are able to reuse the calculations from Example 6.1.4. In particular, we see from Table 6.5 that the only (non-trivial) conjugacy class which fixes anything is (132), which sends:

$$j - l \mapsto (j-k) - (j-l) \qquad j' - l' \mapsto (j'-k') - (j'-l').$$

Therefore the character $\chi_{\widetilde{W}}$ of $\widetilde{\mathcal{WH}}_{(3,1)}$ as an S_4-module is,

	1	(12)	(12)(34)	(123)	(1234)
$\chi_{\widetilde{U}}$	16	0	0	-2	0

A similar calculation to Eq. 6.6 gives the irreducible decomposition,

$$\widetilde{\mathcal{WH}}_{(3,1)} \cong \left(\ydiagram{3,1} \oplus \ydiagram{2,2} \oplus \ydiagram{2,1,1} \right)^{\oplus 2}.$$

As an S_2**-module.** Again, we see that $(12) \in \mathrm{S}_2$ acts on the 0-chains as follows.

$$\begin{array}{cccccc} a \leftrightarrow a' & b \leftrightarrow b' & c \leftrightarrow c' & d \leftrightarrow d' & e \leftrightarrow e' & f \leftrightarrow f' \\ g \leftrightarrow g' & h \leftrightarrow h' & i \leftrightarrow i' & j \leftrightarrow j' & k \leftrightarrow k' & l \leftrightarrow l', \end{array}$$

The same argument as above applies here; the matrix corresponding to (12) is traceless and so the irreducible S_2-module decomposition of the 16 dimensional representation $\widetilde{\mathcal{WH}}_{(3,1)}$ is,

$$\widetilde{\mathcal{WH}}_{(3,1)} \cong \left(\ydiagram{2} \oplus \ydiagram{1,1} \right)^{\oplus 8}$$

As an $(\mathrm{S}_2, \mathrm{S}_4)$**-bimodule.** Similarly to above, we see that the S_2-invariants $\left(\widetilde{\mathcal{WH}}_{(3,1)}\right)^{\mathrm{S}_2}$ have basis,

$$\left(\widetilde{\mathcal{WH}}_{(3,1)}\right)^{\mathrm{S}_2} = \langle (a-b)+(a'-b'), (a-c)+(a'-c'), \dots (j-l)+(j'-l') \rangle,$$

which is 8-dimensional. The only 8-dimensional subrepresentation of the S_4-module $\widetilde{\mathcal{WH}}_{(3,1)}$ is,

$$\ydiagram{3,1} \oplus \ydiagram{2,2} \oplus \ydiagram{2,1,1}\,.$$

Therefore we must have the following decomposition into irreducible $(\mathrm{S}_2, \mathrm{S}_4)$-bimodules:

$$\widetilde{\mathcal{WH}}_{(3,1)} \cong \left(\ydiagram{1,1} \oplus \ydiagram{2} \right) \otimes \left(\ydiagram{3,1} \oplus \ydiagram{2,2} \oplus \ydiagram{2,1,1} \right).$$

Induced module structure on $\widetilde{\mathcal{C}}_\lambda$

Recall that in Lemma 6.1.25 we recovered $\mathcal{C}_\lambda$ as an induced module,

$$\mathcal{C}_\lambda \cong \mathrm{Ind}_{G_\lambda}^{\mathrm{S}_n} \mathcal{C}_x.$$

We proceed analogously here, first defining the stabilizer subgroups $\widetilde{G}_\lambda$.

Lemma 6.2.10. *Fix a partition $\lambda \vdash n$ of length i. For any two elements $\widetilde{x}_1, \widetilde{x}_2 \in \widetilde{\Pi}_\lambda$ there is an isomorphism,*

$$\mathrm{Stab}_{\mathrm{S}_i \times \mathrm{S}_n}(\widetilde{x}_1) \cong \mathrm{Stab}_{\mathrm{S}_i \times \mathrm{S}_n}(\widetilde{x}_2).$$

Proof. There exists an element $\theta \in \mathrm{S}_i \times \mathrm{S}_n$ such that $\theta \cdot \widetilde{x}_1 = \widetilde{x}_2$. As in Lemma 6.1.18, the isomorphism is conjugation by this element. □

Again, this result allows us to make the following definition.

Definition 6.2.11. Fix $\lambda \vdash n$ of length i. Define $\widetilde{G}_\lambda$ to be the stabilizer subgroup,

$$\widetilde{G}_\lambda := \mathrm{Stab}_{\mathrm{S}_i \times \mathrm{S}_n}(\widetilde{x}),$$

for any $\widetilde{x} \in \widetilde{\Pi}_\lambda$.

We seek to describe the subgroups $\widetilde{G}_\lambda$. There is a trivial embedding,

$$G_\lambda \cong 1 \times G_\lambda \hookrightarrow \mathrm{S}_i \times \mathrm{S}_n,$$

where the element $g \in G_\lambda$ is mapped to $(1, g) \in \mathrm{S}_i \times \mathrm{S}_n$. We can therefore ask how G_λ acts on $\widetilde{\Pi}_\lambda$ under this embedding. It is not true in general that G_λ stabilizes $\widetilde{x} \in \widetilde{\Pi}_\lambda$, as it may permute blocks of the same size. Given such an element $g \in G_\lambda$, let $\nu_\lambda(g) \in \mathrm{Sym}(\{B_1, \ldots, B_i\}) \cong \mathrm{S}_i$ denote the permutation of the blocks of $\widetilde{x} \in \widetilde{\Pi}_\lambda$ induced by the action of $g \in G_\lambda$.

We can use $\nu_\lambda(g)$ to define a twisted embedding,

$$\begin{aligned} \Theta_\lambda : G_\lambda &\hookrightarrow \mathrm{S}_i \times \mathrm{S}_n \\ g &\mapsto (\nu_\lambda(g), g). \end{aligned}$$

Untangling this twisted embedding allows us to describe the group structure of $\widetilde{G}_\lambda$.

Proposition 6.2.12. *Fix a partition $\lambda = 1^{m_1} 2^{m_2} \cdots \vdash n$ of length i.*

1. *The image of G_λ under the twisted embedding Θ_λ is isomorphic to $\widetilde{G}_\lambda$. Therefore the embedding Θ_λ induces an isomorphism,*

$$G_\lambda \cong \widetilde{G}_\lambda.$$

2. The stabilizer $\widetilde{G}_\lambda$ is isomorphic to the subgroup of,

$$\prod_j \left(\mathrm{S}_{m_j} \times \mathrm{S}_{m_j}[\mathrm{S}_j]\right),$$

consisting of elements of the form,

$$\prod_j (\sigma_j, [\sigma_j; \underline{\tau}_j]).$$

Proof. Fix an ordered set partition $\widetilde{x} \in \widetilde{\Pi}_\lambda$. By Lemma 6.1.20, we can identify $g \in G_\lambda$ with,

$$\prod_j [\sigma_j, \underline{\tau}_j] \in \prod_j \mathrm{S}_{m_j}[\mathrm{S}_j].$$

The element,

$$(\sigma_1, \sigma_2, \ldots) \in \prod_j \mathrm{S}_{m_j},$$

is identified with the permutation $\nu_\lambda(g)$ on the blocks of $\widetilde{x}$ under the natural inclusion,

$$\prod_j \mathrm{S}_{m_j} \hookrightarrow \mathrm{S}_i\,.$$

Therefore, the twisted embedding factors through $\prod_j \left(\mathrm{S}_{m_j} \times \mathrm{S}_{m_j}[\mathrm{S}_j]\right) \leq \mathrm{S}_i \times \mathrm{S}_n$ as follows:

$$\begin{aligned} G_\lambda \hookrightarrow \prod_j \left(\mathrm{S}_{m_j} \times \mathrm{S}_{m_j}[\mathrm{S}_j]\right) &\hookrightarrow \mathrm{S}_i \times \mathrm{S}_n \\ \prod_j [\sigma_j, \underline{\tau}_j] \mapsto \prod_j \left(\sigma_j, [\sigma_j; \underline{\tau}_j]\right) &\mapsto (\nu_\lambda(g), g), \end{aligned}$$

and we have shown that the image of the twisted embedding is isomorphic to the subgroup of $\prod_j \left(\mathrm{S}_{m_j} \times \mathrm{S}_{m_j}[\mathrm{S}_j]\right)$ consisting of elements of the form $\prod_j \left(\sigma_j, [\sigma_j; \underline{\tau}_j]\right)$. Restrict attention to the j-th factor of $\widetilde{x}$ consisting of m_j blocks of size j,

$$(B_1, \ldots, B_{m_j}).$$

The action of $(\nu, [\sigma; \underline{\tau}]) \in \mathrm{S}_{m_j} \times \mathrm{S}_{m_j}[\mathrm{S}_j]$ on $(B_1, \ldots, B_{m_j})$ is given by,

$$\begin{aligned}\nu \cdot (B_1, \ldots, B_{m_j}) \cdot [\sigma; \underline{\tau}] &= \nu \cdot (B_1, \ldots, B_{m_j}) \cdot [\sigma; \underline{1}] \cdot [1; \underline{\tau}] \\ &= (\nu \cdot B_1 \cdot \sigma, \ldots, \nu \cdot B_{m_j} \cdot \sigma) \cdot [1; \underline{\tau}],\end{aligned}$$

where the right action of σ on B_k is by $B_k \cdot \sigma = \sigma^{-1} \cdot B_k$, and where $\underline{\tau}$ only permutes elements within the blocks and thus acts trivially on $\widetilde{x}$. We see immediately that the only elements that fix $\widetilde{x}$ must satisfy $\nu = \sigma$, as desired. $\square$

Lemma 6.2.13. *Let $\lambda = 1^{m_1} 2^{m_2} \cdots \vdash n$ and let $\widetilde{x} \in \widetilde{\Pi}_\lambda$ of type λ. The complex $\mathcal{C}_{\widetilde{x}}$ is a $\widetilde{G}_\lambda$-module and,*

$$\widetilde{\mathcal{C}}_\lambda \cong \mathrm{Ind}_{\widetilde{G}_\lambda}^{\mathrm{S}_i \times \mathrm{S}_n} \mathcal{C}_{\widetilde{x}}.$$

Proof. This is almost identical to the proof of Lemma 6.1.25. $\square$

Remark 6.2.14. Of particular relevance to what follows is the special case $\lambda = (a^i)$. Let $n = a \cdot i$ and consider $x \in \Pi_n$ of type λ. Concretely, x is a set partition of n into i blocks each of size a. In this case we have that,

$$G_{(a^i)} \cong \mathrm{S}_i[\mathrm{S}_a],$$

and that, $\widetilde{G}_{(a^i)}$ is the subgroup of $\mathrm{S}_i \times \mathrm{S}_i[\mathrm{S}_a]$ consisting of elements of the form $(\sigma, [\sigma; \underline{\tau}])$ where $\sigma \in \mathrm{S}_i$ and $\underline{\tau} \in \mathrm{S}_a^{\times i}$.

S_i-invariants

A simple observation is that we can recover the S_n-modules $\mathcal{WH}_\lambda$ and $\mathcal{WH}_{i,n}$ from their extended counterparts $\widetilde{\mathcal{WH}}_\lambda$ and $\widetilde{\mathcal{WH}}_{i,n}$ by taking S_i-invariants.

Lemma 6.2.15. *Fix a partition $\lambda \vdash n$ of length i. There are S_n-module isomorphisms,*

$$\left(\widetilde{\mathcal{WH}}_\lambda\right)^{\mathrm{S}_i} \cong \mathcal{WH}_\lambda, \qquad \left(\widetilde{\mathcal{WH}}_{i,n}\right)^{\mathrm{S}_i} \cong \mathcal{WH}_{i,n}.$$

Proof. Equivariance of the differentials in $\widetilde{\mathcal{C}}_\lambda$ and $\widetilde{\mathcal{C}}_{i,n}$ means it suffices to exhibit S_n-module isomorphisms,

$$\left(\widetilde{\mathcal{C}}_\lambda\right)^{\mathrm{S}_i} \cong \mathcal{C}_\lambda, \qquad \left(\widetilde{\mathcal{C}}_{i,n}\right)^{\mathrm{S}_i} \cong \mathcal{C}_{i,n}.$$

First note that the S_i-module $\widetilde{\mathcal{C}}_\lambda$ decomposes as a sum of S_i-submodules,

$$\widetilde{\mathcal{C}}_\lambda = \bigoplus_{\substack{x\in\Pi_n \\ \mathrm{type}(x)=\lambda}} A_x,$$

where $A_x := \bigoplus_{\widetilde{x}\in\mathrm{supp}^{-1}(x)} \mathcal{C}_{\widetilde{x}}$ and where the S_i-action is by permuting summands. It is easy to see that the S_i-invariants of A_x can be identified with $\mathcal{C}_x$. Indeed, S_i can be identified with the permutation group of $\mathrm{supp}^{-1}(x)$. Therefore, the S_i-invariants of $\widetilde{\mathcal{C}}_\lambda$ are isomorphic to,

$$\bigoplus_{\substack{x\in\Pi_x \\ \mathrm{type}(x)=\lambda}} \mathcal{C}_x,$$

which is precisely $\mathcal{C}_\lambda$. As usual, apply this argument to $\widetilde{\mathcal{C}}_{i,n}$ componentwise. □

6.2.4 The $(\mathrm{S}_i, \mathrm{S}_n)$-bimodule structure of $\widetilde{\mathcal{WH}}_\lambda$ in terms of the twisted Lie operad

In this section we describe the decomposition of the extended type-selected Whitney homology $\widetilde{\mathcal{WH}}_\lambda$ into irreducible $(\mathrm{S}_i, \mathrm{S}_n)$-bimodules. Recall that irreducible $(\mathrm{S}_i, \mathrm{S}_n)$-bimodules are of the form,

$$P_\mu \otimes P_\nu,$$

for partitions $\mu \vdash i$ and $\nu \vdash n$. Before presenting the main theorem, we introduce some notation.

Definition 6.2.16. Fix $i, n \in \mathbb{N}$. Let $\lambda \vdash n$ of length i and with m_j parts of length j, and let $\mu \vdash i$.

1. Define a λ-tuple of μ to be a tuple,

$$\underline{\mu} := (\mu_1, \mu_2, \ldots),$$

where $\mu_j \vdash m_j$. Let $\Lambda_\lambda(\mu)$ denote the set of all λ-tuples of μ. Notice that $|\Lambda_\lambda(\mu)| < \infty$.

2. Given a λ-tuple of μ, define its even-conjugate,

$$\underline{\mu}^\vee := (\mu_1, \mu_2', \ldots).$$

That is, conjugate every even indexed partition. Note that $\underline{\mu}^\vee$ is still a λ-tuple of μ.

3. Given a λ-tuple of μ, let,

$$P_{\underline{\mu}} := P_{\mu_1} \otimes P_{\mu_2} \otimes \cdots,$$

denote the irreducible $\mathrm{S}_{m_1} \times \mathrm{S}_{m_2} \times \cdots$-module indexed by $\underline{\mu}$.

4. Let $a_{\underline{\mu}}$ denote the multiplicity with which $P_{\underline{\mu}}$ appears in the restriction,

$$\mathrm{Res}^{\mathrm{S}_i}_{\mathrm{S}_{m_1} \times \mathrm{S}_{m_2} \times \cdots} P_\mu.$$

That is, we have,

$$\mathrm{Res}^{\mathrm{S}_i}_{\mathrm{S}_{m_1} \times \mathrm{S}_{m_2} \times \cdots} P_\mu = \bigoplus_{\underline{\mu}} a_{\underline{\mu}} P_{\underline{\mu}}.$$

Call a λ-partition $\underline{\mu}$ positive if $a_{\underline{\mu}} \geq 0$.

5. Let $\underline{\mu}$ be a λ-tuple of μ. Denote by,

$$P_{\underline{\mu}} \left[\!\left[\widehat{\mathsf{Lie}} \right]\!\right],$$

the representation,

$$\left[(P_{\mu_1} \circ \widehat{\mathsf{Lie}}_1) \circledast (P_{\mu_2} \circ \widehat{\mathsf{Lie}}_2) \circledast \cdots \right],$$

and define

$$P_{\mu}^{\vee}\left[\!\left[\widehat{\mathsf{Lie}}\right]\!\right]_{\lambda} := \bigoplus_{\underline{\mu}\in\Lambda_{\lambda}(\mu)} a_{\underline{\mu}} P_{\underline{\mu}^{\vee}}\left[\!\left[\widehat{\mathsf{Lie}}\right]\!\right].$$

We compute some examples to help parse all of these definitions.

Example 6.2.17.

1. Let $\lambda = (2, 2) \vdash 4$ of length $i = 2$. There are only two partitions $\mu \vdash i = 2$. We have that $m_2 = 2$ and so any λ-tuple of μ must be of the form,

   $$\underline{\mu} = (\varnothing, \mu_2, \varnothing, \cdots),$$

   where $\mu_2 \vdash m_2 = 2$.

   (a) Consider the partition $\mu = (2)$. The only positive λ-tuple of μ is,

   $$\underline{\mu} = (\varnothing, (2), \varnothing, \cdots),$$

   for which $a_{\underline{\mu}} = 1$, and the corresponding summand, $P_{\underline{\mu}^{\vee}}\left[\!\left[\widehat{\mathsf{Lie}}\right]\!\right]$ is,

   $$P_{\mu_2'} \circ \widehat{\mathsf{Lie}}_2 = \ydiagram{1,1} \circ \ydiagram{2} = \ydiagram{3,1}.$$

   (b) Consider the partition $\mu = (1, 1)$. The only positive λ-tuple of μ is,

   $$\underline{\mu} = (\varnothing, (1, 1), \varnothing, \cdots),$$

   where again $a_{\underline{\mu}} = 1$. The corresponding summand, $P_{\underline{\mu}^{\vee}}\left[\!\left[\widehat{\mathsf{Lie}}\right]\!\right]$ is,

   $$P_{\mu_2'} \circ \widehat{\mathsf{Lie}}_2 = \ydiagram{2} \circ \ydiagram{2} = \ydiagram{2,2} \oplus \ydiagram{4}.$$

2. Let $\lambda = (3, 1) \vdash 4$ of length $i = 2$. Again there are only two partitions $\mu \vdash i$ to consider. Now $m_1 = m_3 = 1$ and so any λ-tuple must be of the form,

   $$\underline{\mu} = (\mu_1, \varnothing, \mu_3, \varnothing, \cdots),$$

where $\mu_1 \vdash m_1 = 1$ and $\mu_3 \vdash m_3 = 1$. In fact, we are forced to have,

$$\underline{\mu} = ((1), \varnothing, (1), \varnothing, \cdots),$$

regardless of μ.

(a) Consider the partition $\mu = (2)$. The only λ-tuple,

$$\underline{\mu} = ((1), \varnothing, (1), \varnothing, \cdots),$$

is positive, with $a_{\underline{\mu}} = 1$. This has corresponding summand, $P_{\underline{\mu}^\vee}\left\llbracket \widehat{\mathsf{Lie}} \right\rrbracket$,

$$\left(P_{\mu_1} \circ \widehat{\mathsf{Lie}}_1\right) \circledast \left(P_{\mu_3} \circ \widehat{\mathsf{Lie}}_3\right) = \ydiagram{1} \circledast \ydiagram{2,1} = \ydiagram{3,1} \oplus \ydiagram{2,2} \oplus \ydiagram{2,1,1}.$$

(b) Consider $\mu = (1,1)$. The exact same calculation above applies here too, and and we get that, the summand $P_{\underline{\mu}^\vee}\left\llbracket \widehat{\mathsf{Lie}} \right\rrbracket$, corresponding to $\mu = (1,1)$ is,

$$\left(P_{\mu_1} \circ \widehat{\mathsf{Lie}}_1\right) \circledast \left(P_{\mu_3} \circ \widehat{\mathsf{Lie}}_3\right) = \ydiagram{1} \circledast \ydiagram{2,1} = \ydiagram{3,1} \oplus \ydiagram{2,2} \oplus \ydiagram{2,1,1}.$$

Remark 6.2.18. The representations appearing here corresponding to $\lambda = (2,2)$ and $\lambda = (3,1)$ may look familiar to those computed in Example 6.2.3. Concretely, we showed that,

$$\widetilde{\mathcal{WH}}_{(2,2)} \cong \ydiagram{1,1} \otimes \left(\ydiagram{4} \oplus \ydiagram{2,2} \right) \bigoplus \ydiagram{2} \otimes \ydiagram{3,1}.$$

and that,

$$\widetilde{\mathcal{WH}}_{(3,1)} \cong \left(\ydiagram{1,1} \oplus \ydiagram{2} \right) \otimes \left(\ydiagram{3,1} \oplus \ydiagram{2,2} \oplus \ydiagram{2,1,1} \right).$$

We point out that the not only do the same irreducible S_4-modules appear, they also appear paired with the same μ partitions.

We are now ready to present the main theorem of this section describing the $\mathrm{S}_i \times \mathrm{S}_n$-module structure of the extended Whitney homology of bidegree (i, n).

Theorem 6.2.19. *Fix $i, n \in \mathbb{N}$ and let $\lambda \vdash n$ be a partition of length i. There is an isomorphism of $(\mathrm{S}_i, \mathrm{S}_n)$-bimodules,*

$$\widetilde{\mathcal{WH}}_\lambda \cong \bigoplus_{\mu \vdash i} P_\mu \otimes P_\mu^\vee \left[\!\left[\widehat{\mathsf{Lie}} \right]\!\right]_\lambda .$$

Proof. Let $\lambda = 1^{m_1} 2^{m_2} \cdots \vdash n$ and $x \in \Pi_n$ of type λ. By Lemma 6.1.10, the interval poset $(\perp, x)$ splits as a product of posets,

$$(\perp, x) \cong \prod_j \left(\Pi_j^{\times m_j} \right).$$

It will thus suffice to consider type λ where λ is of the form $(a^i) \vdash n$, and thus,

$$(\perp, x) \cong \Pi_a \times \cdots \times \Pi_a .$$

By Lemma 6.1.24 we have that the top homology $H_*(\perp, x)$ of the interval $(\perp, x)$ is an $\mathrm{S}_i[\mathrm{S}_a]$-module isomorphic to $\mathcal{R}_i[\pi_a]$ where,

$$\mathcal{R}_i \cong \begin{cases} P_{[i]} & a \text{ odd} \\ P_{[1^i]} & a \text{ even} \end{cases} .$$

Further, by Lemma 6.2.13, we have that $\mathcal{R}_i[\pi_a]$ is a $\widetilde{G}_\lambda$-module and that,

$$\widetilde{\mathcal{WH}}_\lambda \cong \mathrm{Ind}_{\widetilde{G}_\lambda}^{\mathrm{S}_i \times \mathrm{S}_n} \mathcal{R}_i[\pi_a].$$

As an intermediate step, let U be the $\mathrm{S}_i \times \mathrm{S}_i[\mathrm{S}_a]$-module,

$$\mathrm{Ind}_{\widetilde{G}_\lambda}^{\mathrm{S}_i \times \mathrm{S}_i[\mathrm{S}_a]} \mathcal{R}_i[\pi_a],$$

so that $\widetilde{\mathcal{WH}}_\lambda \cong \mathrm{Ind}_{\mathrm{S}_i \times \mathrm{S}_i[\mathrm{S}_a]}^{\mathrm{S}_i \times \mathrm{S}_n} U$.

Note that the cosets $\mathrm{S}_i \times \mathrm{S}_i[\mathrm{S}_a]/\widetilde{G}_\lambda$ can be identified with S_i, and so we have that U is isomorphic as a vector space to,

$$\Bbbk[\mathrm{S}_i] \otimes \mathcal{R}_i[\pi_a].$$

It remains to describe the $\mathrm{S}_i \times \mathrm{S}_i[\mathrm{S}_a]$-module structure of U. Let, $(\sigma, [\eta; \underline{\tau}]) \in \mathrm{S}_i \times \mathrm{S}_i[\mathrm{S}_a]$ and $x \otimes y \in \Bbbk[\mathrm{S}_i] \otimes \mathcal{R}_i[\pi_a]$ where $x \in \mathrm{S}_i$ and $y = r \otimes p_1 \cdots p_i \in \mathcal{R}_i[\pi_a]$. Then we have the following explicit description of the action.

$$\begin{aligned}
(\sigma, [\eta; \underline{\tau}]) \cdot (x \otimes y) &= (\sigma x, [\eta; \underline{\tau}])(1 \otimes y) \\
&= (\sigma x \eta^{-1}, 1_{\mathrm{S}_i[\mathrm{S}_a]})(\eta, [\eta; \underline{\tau}])(1 \otimes y) \\
&= (\sigma x \eta^{-1}, (\eta, [\eta; \underline{\tau}]) \cdot y)
\end{aligned} \tag{6.15}$$

where $1_{\mathrm{S}_i[\mathrm{S}_a]}$ is the identity in $\mathrm{S}_i[\mathrm{S}_a]$, and where $(\eta, [\eta; \underline{\tau}]) \cdot y$ is determined by the usual $\widetilde{G}_\lambda$-module structure on $\mathcal{R}_i[\pi_a]$.

Recall the well-known decomposition of the group algebra $\Bbbk[\mathrm{S}_i]$,

$$\Bbbk[\mathrm{S}_i] \cong \bigoplus_{\mu \vdash i} P_\mu \otimes P_\mu,$$

as an $(\mathrm{S}_i, \mathrm{S}_i)$-bimodule. Concretely, the group $\mathrm{S}_i \times \mathrm{S}_i$ acts on $\Bbbk[\mathrm{S}_i]$ where the left copy, consisting of elements of the form $(\sigma, 1)$, of S_i acts by left translation, and the right copy of, consisting of elements of the form $(1, \eta)$, of S_i by right translation. Under the decomposition above, elements of the form $(\sigma, 1)$ act on $P_\mu \otimes P_\mu$ sending elements (p, q) to $(\sigma \cdot p, q)$ and elements of the form $(1, \eta)$ act on $P_\mu \otimes P_\mu$ sending elements (p, q) to $(p, q \cdot \eta)$.

We recognize one copy of S_i sitting in $\mathrm{S}_i \times \mathrm{S}_i[\mathrm{S}_a]$ consisting of elements of the form $(\sigma, 1)$ for $\sigma \in \mathrm{S}_i$. From Eq. 6.15 this is seen to act on $\Bbbk[\mathrm{S}_i]$ by left translation:

$$(\sigma, 1) \cdot (x \otimes y) = (\sigma x, y).$$

Similarly, we recognize another copy of S_i sitting in $\mathrm{S}_i \times \mathrm{S}_i[\mathrm{S}_a]$ consisting of elements of the form $(1, [\eta; 1])$ for $\eta \in \mathrm{S}_i$. From Eq. 6.15 this is seen to act on $\Bbbk[\mathrm{S}_i]$ simultaneously by right translation on $\Bbbk[\mathrm{S}_i]$ and by the usual action of $\widetilde{G}_\lambda$ on $\mathcal{R}_i[\pi_a]$:

$$(1, [\eta; 1]) \cdot (x \otimes y) = (x \cdot \eta^{-1}, (\eta, [\eta; 1]) \cdot y).$$

This is summarized by the following isomorphism of $\mathrm{S}_i \times \mathrm{S}_i[\mathrm{S}_a]$-modules,

$$U \cong \bigoplus_{\mu \vdash i} P_\mu \otimes P_\mu \otimes_{\mathrm{S}_i} \mathcal{R}_i[\pi_a].$$

When a is odd, $\mathcal{R}_i$ is the one-dimensional trivial representation of S_i, and we have that,

$$P_\mu \otimes_{\mathrm{S}_i} \mathcal{R}_i[\pi_a] \cong P_\mu \otimes_{\mathrm{S}_i} \pi_a^{\otimes i} \cong \mathbb{S}_\mu(\pi_a).$$

When a is even, $\mathcal{R}_i$ is the one-dimensional sign representation of S_i. This twists the action of S_i on $\pi_a^{\otimes i}$ by the sign and we have that,

$$P_\mu \otimes_{\mathrm{S}_i} \mathcal{R}_i[\pi_a] \cong P_{\mu'} \otimes_{\mathrm{S}_i} \pi_a^{\otimes i} \cong \mathbb{S}_{\mu'}(\pi_a),$$

where $\mu' \vdash i$ is the conjugate partition of μ. The result follows from Proposition 6.1.15 which identifies π_a with $\epsilon_a \otimes \mathsf{Lie}_a = \widehat{\mathsf{Lie}}_a$. $\square$

Example 6.2.20. Putting Examples 6.2.3 and 6.2.17 together we see the theorem in action.

- $\lambda = (2, 2)$

  - $\mu = (2)$

$$P_{(2)} \otimes P_{(2)}^\vee \left[\!\left[\widehat{\mathsf{Lie}}\right]\!\right]_{(2,2)} = \ydiagram{2} \otimes \ydiagram{3,1}$$

  - $\mu = (1, 1)$

$$P_{(1,1)} \otimes P_{(1,1)}^\vee \left[\!\left[\widehat{\mathsf{Lie}}\right]\!\right]_{(2,2)} = \ydiagram{1,1} \otimes \left(\ydiagram{4} \oplus \ydiagram{2,2} \right)$$

Theorem 6.2.19 gives us,

$$\widetilde{\mathcal{WH}}_{(2,2)} \cong \ydiagram{1,1} \otimes \left(\ydiagram{4} \oplus \ydiagram{2,2} \right) \quad \bigoplus \quad \ydiagram{2} \otimes \ydiagram{3,1}\,.$$

as expected.

- $\lambda = (3,1)$

  - $\mu = (2)$

$$P_{(2)} \otimes P^{\vee}_{(2)} \left\llbracket \widehat{\mathsf{Lie}} \right\rrbracket_{(3,1)} = \ydiagram{2} \otimes \left(\ydiagram{3,1} \oplus \ydiagram{2,2} \oplus \ydiagram{2,1,1} \right)$$

  - $\mu = (1,1)$

$$P_{(1,1)} \otimes P^{\vee}_{(1,1)} \left\llbracket \widehat{\mathsf{Lie}} \right\rrbracket_{(3,1)} = \ydiagram{1,1} \otimes \left(\ydiagram{3,1} \oplus \ydiagram{2,2} \oplus \ydiagram{2,1,1} \right)$$

Theorem 6.2.19 gives us,

$$\widetilde{\mathcal{WH}}_{(3,1)} \cong \left(\ydiagram{1,1} \oplus \ydiagram{2} \right) \otimes \left(\ydiagram{3,1} \oplus \ydiagram{2,2} \oplus \ydiagram{2,1,1} \right).$$

as expected.

Remark 6.2.21. In Lemma 6.2.15 we showed that the S_i-invariants of $\widetilde{\mathcal{WH}}_{\lambda}$ coincide with $\mathcal{WH}_{\lambda}$. Together with the description given in Theorem 6.2.19 we have that,

$$\mathcal{WH}_{\lambda} \cong \left(\bigoplus_{\mu \vdash i} P_{\mu} \otimes P^{\vee}_{\mu} \left\llbracket \widehat{\mathsf{Lie}} \right\rrbracket_{\lambda} \right)^{\mathrm{S}_i} \cong P^{\vee}_{(i)} \left\llbracket \widehat{\mathsf{Lie}} \right\rrbracket_{\lambda}$$

This agrees with the descriptions given in [18] Section 2.3. In particular, in [18] Eq. (26) they state (albeit in the language of characters) that,

$$\mathcal{WH}_{\lambda} \cong \underbrace{\left(\circledast P_{(m_j)} \circ \widehat{\mathsf{Lie}}_j \right)}_{j \text{ odd}} \circledast \underbrace{\left(\circledast P_{(1^{m_j})} \circ \widehat{\mathsf{Lie}}_j \right)}_{j \text{ even}}.$$

This follows from taking the S_i-invariants of Theorem 6.2.19 after noting that there is only one positive λ-tuple of $\mu = (i)$, namely,

$$\underline{\mu} = ((m_1), (m_2), \ldots),$$

and it satisfies $a_{\underline{\mu}} = 1$.

6.3 Extended Whitney homology as a PD-module

In this section we realise the extended Whitney homology as a PD-module, and as such are able to apply the theory developed in Chapter 5 to attain representation stability results for the coefficients $c_{\lambda\mu}$. Moreover, we recover stability results in [18] relating to the usual Whitney homology.

6.3.1 Restriction of a set partition

Throughout this section we will make the following notational conventions. Let $\lambda = 1^{m_1}2^{m_2}\cdots \vdash n$ be a partition of n into i blocks. Let $m := m_1$ be the number of parts of size 1 in λ. Let $k = n - m$ and $l = i - m$.

Definition 6.3.1. Define the restriction of λ as the partition

$$\lambda_\circ = 2^{m_2}3^{m_3}\cdots \vdash k.$$

obtained from λ by removing all m parts of size 1. Note that $\lambda_\circ$ is of length l.

Continuing to establish notation, let $\widetilde{x} \in \widetilde{\Pi}_n$ be of type λ. Assume that $\widetilde{x}$ contains (at least) one singleton block B. Then there is a canonical way to remove the singleton B and obtain an ordered set partition $\widetilde{x}_{-B} \in \Pi_{n-1}$ by first deleting the block B, and then by relabelling all elements j in $\widetilde{x}$ greater than the element in B by $j-1$. For example,

$$\widetilde{x} = 89 \parallel 23 \parallel 4 \parallel 567 \parallel 1 \quad \rightsquigarrow \quad \widetilde{x}_{-\{4\}} = 78 \parallel 23 \parallel 456 \parallel 1.$$

Repeating this as necessary[2], there is a canonical way to remove *all* singleton blocks, and obtain an ordered set partition $\widetilde{x}_\circ \in \widetilde{\Pi}_k$ of length l. Concretely, given an ordered set partition $\widetilde{x} \in \widetilde{\Pi}_n$, let,

$$\mathrm{Sing}(\widetilde{x}) := \bigsqcup_{\substack{B \in \widetilde{x} \\ |B|=1}} B,$$

and define a map,

$$\begin{aligned} \phi : \mathbf{n} &\to \mathbf{k} \\ j &\mapsto j - \rho(j) \end{aligned}$$

where $\rho(j) = \#\{a \in \mathrm{Sing}(\widetilde{x}) : a \leq j\}$.

Definition 6.3.2. Given an ordered set partition $\widetilde{x} \in \widetilde{\Pi}_\lambda$, its restriction $\widetilde{x}_\circ \in \widetilde{\Pi}_k$ is obtained from $\widetilde{x}$ by removing blocks of size 1 and relabelling the remaining elements with the function ϕ.

Example 6.3.3. Let $\widetilde{x} = 89 \parallel 23 \parallel 4 \parallel 567 \parallel 1 \in \widetilde{\Pi}_9$ of type $(3,2,2,1,1)$. Then,

$$\widetilde{x} = 89 \parallel 23 \parallel 4 \parallel 567 \parallel 1 \quad \rightsquigarrow \quad \widetilde{x}_\circ = 67 \parallel 12 \parallel 345.$$

and $\widetilde{x}_\circ \in \Pi_7$ is of type $(3,2,2)$.

Comparing complexes $\mathcal{C}_{\widetilde{x}}$ and $\mathcal{C}_{\widetilde{x}_\circ}$ The operation of removing singleton blocks does not affect the vector space structure of the induced chain complexes.

[2]Note that the order the singletons are removed is immaterial.

Lemma 6.3.4. *Let $\widetilde{x} \in \widetilde{\Pi}_n$ be an ordered set partition containing a singleton block B, and let $\widetilde{y} := \widetilde{x}_{-B} \in \Pi_{n-1}$ denote the ordered set partition obtained from $\widetilde{x}$ by removing the block B. Then there is an isomorphism of complexes,*

$$\mathcal{C}_{\widetilde{x}} \cong \mathcal{C}_{\widetilde{y}}.$$

Furthermore, there is an isomorphism of complexes,

$$\mathcal{C}_{\widetilde{x}} \cong \mathcal{C}_{\widetilde{x}_\circ}.$$

Proof. The second assertion follows by (possible) repeated applications of the first. The complex $\mathcal{C}_{\widetilde{x}}$ is the order complex of the interval $(\perp, x) \leq \Pi_n$ where $x = \mathrm{supp}(\widetilde{x})$. Note that any set partition $z \in (\perp, x)$ is a refinement of x, and, since singleton blocks cannot be refined, we have that z must also contain the singleton B. Removing this block from z gives a set partition $z_{-B} \in (\perp, y)$, where $y := \mathrm{supp}(\widetilde{y})$. Note that,

$$w < z \iff w_{-B} < z_{-B},$$

for $w, z \in (\perp, x)$.

Going in the other direction, given a set partition $z' \in (\perp, y)$, it is possible to add the block B and obtain a set partition $z'_{+B} \in (\perp, x)$. Concretely, the set partition z'_{+B} is obtained from z' be first adding B and then relabelling elements j in z' larger than the element in B by $j+1$. This is clearly inverse to the $(-)_{-B}$ operation, and it is also seen to be order preserving. We have exhibited a poset isomorphism,

$$(\perp, x) \cong (\perp, y),$$

and this completes the proof. $\square$

This is an isomorphism on the level of vector spaces. We are able to relate the $\widetilde{G}_\lambda$-module structure of $\mathcal{C}_{\widetilde{x}}$ to the $\widetilde{G}_{\lambda_\circ}$-module structure of $\mathcal{C}_{\widetilde{x}_\circ}$. First we compare the respective stabilizer groups.

Lemma 6.3.5. *Let $\lambda \vdash n$ be a partition with m parts of size 1. There are group isomorphisms,*

1. $G_\lambda \cong G_{\lambda_\circ} \times \mathrm{S}_m$, *and*
2. $\widetilde{G}_\lambda \cong G_{\lambda_\circ} \times H$, *where* $H = \{(\sigma, \sigma) : \sigma \in \mathrm{S}_m\} \leq \mathrm{S}_i \times \mathrm{S}_i$.

where H is the subgroup of $\mathrm{S}_m \times \mathrm{S}_m$ consisting of elements of the form (σ, σ).

Proof. As usual, declare that λ has m_j parts of size j. Recall that $G_\lambda \cong \prod_j \mathrm{S}_{m_j}[\mathrm{S}_j]$. The first isomorphism is nothing but observing that the summand corresponding to $j = 1$ satisfies,

$$\mathrm{S}_{m_1}[\mathrm{S}_1] \cong \mathrm{S}_m,$$

while the remaining summands are precisely $G_{\lambda_\circ}$.

For the second isomorphism, note that H is the image of S_m under the diagonal embedding,

$$\begin{aligned} d : \mathrm{S}_m &\to \mathrm{S}_m \times \mathrm{S}_m \\ \sigma &\mapsto (\sigma, \sigma). \end{aligned} \tag{6.16}$$

Recall that Φ_μ denotes the twisted embedding of type μ. We have the following diagram of isomorphisms.

$$\begin{tikzcd} \widetilde{G}_\lambda \arrow[r, dashed, "2."] \arrow[d, "\Phi_\lambda^{-1}"'] & \widetilde{G}_{\lambda_\circ} \times H \\ G_\lambda \arrow[r, "1."'] & G_{\lambda_\circ} \times \mathrm{S}_m \arrow[u, "\Phi_{\lambda_\circ} \times d"'] \end{tikzcd}$$

Concretely, the map denoted by 2. sends the element $\prod_j(\sigma_j, [\sigma_j; \underline{\tau}_j])$ to

$$\left(\times_{j\geq 2}(\sigma_j, [\sigma_j; \underline{\tau}_j])\right) \times (\sigma_1, \sigma_1).$$

$\square$

Lemma 6.3.6. *Fix a partition $\lambda \vdash n$. The $\widetilde{G}_{\lambda_\circ}$-module $\mathcal{C}_{\widetilde{x}_\circ}$ can be promoted to a $\widetilde{G}_\lambda$-module, and as such is isomorphic to $\mathcal{C}_{\widetilde{x}}$.*

Proof. In order to promote the $\widetilde{G}_{\lambda_\circ}$-module $\mathcal{C}_{\widetilde{x}_\circ}$ to a $\widetilde{G}_\lambda$-module, we write,

$$\widetilde{G}_\lambda \cong G_{\lambda_\circ} \times H,$$

in the notation of Lemma 6.3.5, and thus it suffices to describe how H acts on $\mathcal{C}_{\widetilde{x}_\circ}$. We see that H acts trivially on $\mathcal{C}_{\widetilde{x}}$. Let $\Bbbk$ be the 1-dimensional trivial H-module. We have the following isomorphisms,

$$\mathcal{C}_{\widetilde{x}_\circ} \cong \mathcal{C}_{\widetilde{x}_\circ} \otimes \Bbbk \cong \mathcal{C}_{\widetilde{x}},$$

where the first isomorphism is on the level of vector spaces and the second is as $\widetilde{G}_\lambda$-modules. $\square$

We can combine this result with Lemma 6.2.13 to compare the type selected order complex $\widetilde{\mathcal{C}}_\lambda$ with the complex associated to its restriction $\mathcal{C}_{\widetilde{x}_\circ}$ where $\widetilde{x} \in \widetilde{\Pi}_\lambda$.

Proposition 6.3.7. *Fix a partition $\lambda \vdash n$ and let $\widetilde{x} \in \widetilde{\Pi}_\lambda$. There is an isomorphism of $\mathrm{S}_i \times \mathrm{S}_n$-modules,*

$$\widetilde{\mathcal{WH}}_\lambda = \mathrm{Ind}_{(\mathrm{S}_l \times \mathrm{S}_k)\times(\mathrm{S}_m \times \mathrm{S}_m)}^{\mathrm{S}_i \times \mathrm{S}_n} \widetilde{\mathcal{WH}}_{\lambda_\circ} \otimes \Bbbk[\mathrm{S}_m].$$

Proof. By Lemma 6.2.13 we have that,

$$\widetilde{\mathcal{C}}_\lambda \cong \mathrm{Ind}_{\widetilde{G}_\lambda}^{\mathrm{S}_i \times \mathrm{S}_n} \mathcal{C}_{\widetilde{x}}.$$

Applying Lemma 6.3.6 gives,

$$\begin{aligned}\widetilde{\mathcal{C}}_\lambda &\cong \mathrm{Ind}_{\widetilde{G}_{\lambda_\circ} \times H}^{\mathrm{S}_i \times \mathrm{S}_n} \mathcal{C}_{\widetilde{x}} \otimes \Bbbk \\ &\cong \mathrm{Ind}_{(\mathrm{S}_l \times \mathrm{S}_k)\times(\mathrm{S}_m \times \mathrm{S}_m)}^{\mathrm{S}_i \times \mathrm{S}_n} \left(\mathrm{Ind}_{\widetilde{G}_{\lambda_\circ} \times H}^{(\mathrm{S}_l \times \mathrm{S}_k)\times(\mathrm{S}_m \times \mathrm{S}_m)} \mathcal{C}_{\widetilde{x}} \otimes \Bbbk \right),\end{aligned}$$

where the last isomorphism is just the standard composition of the induction operation. Finally, the term in the parenthesis can be re-written as follows.

$$\mathrm{Ind}_{\widetilde{G}_{\lambda_\circ}\times H}^{(\mathrm{S}_l\times\mathrm{S}_k)\times(\mathrm{S}_m\times\mathrm{S}_m)}\mathcal{C}_{\widetilde{x}_\circ}\otimes\Bbbk\cong\mathrm{Ind}_{\widetilde{G}_{\lambda_\circ}}^{\mathrm{S}_l\times\mathrm{S}_k}\mathcal{C}_{\widetilde{x}_\circ}\otimes\mathrm{Ind}_H^{\mathrm{S}_m\times\mathrm{S}_m}\Bbbk\cong\widetilde{\mathcal{C}}_{\lambda_\circ}\otimes\Bbbk[\mathrm{S}_m],$$

and the result follows upon taking homology.

$\square$

Example 6.3.8. We can use this to compute $\widetilde{\mathcal{WH}}_{(3,1)}$ as an $(\mathrm{S}_2,\mathrm{S}_4)$-bimodule from the $(\mathrm{S}_1,\mathrm{S}_3)$-bimodule $\widetilde{\mathcal{WH}}_{(3)}$.

We have from Theorem 6.2.19 that $\widetilde{\mathcal{WH}}_{(3)}\cong P_{(1)}\otimes\widehat{\mathsf{Lie}}_3$, i.e.,

$$\widetilde{\mathcal{WH}}_{(3)}\cong\ydiagram{1}\otimes\ydiagram{2,1}\,.$$

Proposition 6.3.7 gives that the extended Whitney homology $\widetilde{\mathcal{WH}}_{(3,1)}$ is isomorphic to,

$$\begin{aligned}\mathrm{Ind}_{(\mathrm{S}_1\times\mathrm{S}_3)\times(\mathrm{S}_1\times\mathrm{S}_1)}^{\mathrm{S}_2\times\mathrm{S}_4}\widetilde{\mathcal{WH}}_{(3)}\otimes\Bbbk[\mathrm{S}_1]&\cong\mathrm{Ind}_{(\mathrm{S}_1\times\mathrm{S}_3)\times(\mathrm{S}_1\times\mathrm{S}_1)}^{\mathrm{S}_2\times\mathrm{S}_4}\left(\ydiagram{1}\otimes\ydiagram{2,1}\right)\otimes\left(\ydiagram{1}\otimes\ydiagram{1}\right).\\&\cong\mathrm{Ind}_{\mathrm{S}_1\times\mathrm{S}_1}^{\mathrm{S}_2}\left(\ydiagram{1}\otimes\ydiagram{1}\right)\otimes\mathrm{Ind}_{\mathrm{S}_3\times\mathrm{S}_1}^{\mathrm{S}_4}\left(\ydiagram{2,1}\otimes\ydiagram{1}\right).\\&\cong\left(\ydiagram{2}\oplus\ydiagram{1,1}\right)\otimes\left(\ydiagram{3,1}\oplus\ydiagram{2,2}\oplus\ydiagram{2,1,1}\right),\end{aligned}$$

which agrees with our previous calculations.

6.3.2 Extended Whitney homology as a PD-module

In this section we restrict our attention to the skeleton of PD whose objects are given by pairs (i,n) (see Remark 5.1.2).

Theorem 6.3.9. *There is a* PD*-module* $\widetilde{\mathcal{WH}}_\bullet$ *taking* $(i,n)\mapsto\widetilde{\mathcal{WH}}_{i,n}$.

Proof. We will show that there is a functor,

$$\widetilde{\mathcal{C}}_\bullet : \mathsf{PD} \to \mathsf{dgVect},$$

taking (i, n) to the complex $\widetilde{\mathcal{C}}_{i,n}$. The result will follow upon taking homology.

A diagonal morphism, $\Delta \in \mathrm{Hom}_{\mathsf{PD}}((i, n), (i + m, n + m))$, consists of the data,

1. $\alpha : \mathbf{i} \hookrightarrow \mathbf{i} + \mathbf{m}$
2. $\beta : \mathbf{n} \hookrightarrow \mathbf{n} + \mathbf{m}$
3. $\gamma : \alpha^C \to \beta^C$ a bijection.

To such a morphism we need to assign, in a functorial manner, a map,

$$\widetilde{\mathcal{C}}_\Delta : \widetilde{\mathcal{C}}_{i,n} \to \widetilde{\mathcal{C}}_{i+m,n+m}.$$

Let $\widetilde{x}$ be an ordered set partition of n of length i so that $\mathcal{C}_{\widetilde{x}}$ is a summand in $\widetilde{\mathcal{C}}_{i,n}$. Write $\widetilde{x} = (B_1, \ldots, B_i)$. Then Δ defines a map,

$$\widetilde{x} = (B_1, \ldots, B_i) \mapsto (B'_1, \ldots, B'_{i+m}) =: \widetilde{y},$$

as follows. The injection α sends the blocks B_i of $\widetilde{x}$ to blocks $B'_{\alpha(i)}$. In addition, the elements of the blocks are permuted according to the injection β. This leaves missing blocks corresponding to α^C, and missing elements corresponding to β^C. The bijection γ gives a rule for filling these missing blocks with singletons. Explicitly, $B'_{\alpha(i)} = \beta(B_i)$ and those B'_i for $i \in \alpha^C$ are singleton blocks filled with elements of β^C determined by the bijection γ. This map is best understood by way of example (see Example 6.3.10).

By Lemma 6.3.4, we have that $\mathcal{C}_{\widetilde{x}} \cong \mathcal{C}_{\widetilde{y}}$, and the map $\widetilde{\mathcal{C}}_\Delta$ restricted to $\mathcal{C}_{\widetilde{x}}$ is defined to be this vector space isomorphism. Preservation of the identity and of composition are routine. $\square$

Example 6.3.10. Let $\Delta = (\alpha, \beta, \gamma) \in \mathrm{Hom}_{\mathsf{PD}}((3,7),(5,9))$, where,

$$\alpha = \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 5 \end{cases} \qquad \beta = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 7 \\ 3 \mapsto 8 \\ 4 \mapsto 2 \\ 5 \mapsto 9 \\ 6 \mapsto 3 \\ 7 \mapsto 5 \end{cases} \qquad \gamma = \begin{cases} 1 \leftrightarrow 6 \\ 4 \leftrightarrow 4 \end{cases}$$

Let $\widetilde{x} = (236|14|57)$ be an ordered set partition in of length $i = 3$ in the extended Whitney complex of $n = 7$. Then Δ determines a map sending $\widetilde{x}$ to,

$$(6|\beta(2)\beta(3)\beta(6)|\beta(1)\beta(4)|4|\beta(5)\beta(7)) = (6|783|12|4|95),$$

which we call $\Delta(x) \in \Pi_9$. This determines a map $\mathcal{C}_{\widetilde{x}} \mapsto \mathcal{C}_{\Delta(\widetilde{x})}$

6.3.3 Stability of the PD-module $\widetilde{\mathcal{WH}}_{\bullet}$

Proposition 6.3.11. *Fix a partition $\lambda \vdash n$ of length i. There is an isomorphism of $(\mathrm{S}_i, \mathrm{S}_n)$-modules,*

$$\widetilde{\mathcal{WH}}_{\lambda} \cong \mathrm{Ind}_{\mathsf{PD}}(\widetilde{\mathcal{WH}}_{\lambda_\circ})_{i,n}.$$

Proof. This is just a restatement of Proposition 6.3.7. □

The PD-module $\widetilde{\mathcal{WH}}$ decomposes into two PD-submodules,

$$\widetilde{\mathcal{WH}} = \widetilde{\mathcal{WH}}^{+} \oplus \widetilde{\mathcal{WH}}^{-},$$

where,

$$\widetilde{\mathcal{WH}}^{+}_{i,n} = \bigoplus_{\lambda = \lambda_\circ} \widetilde{\mathcal{WH}}_{\lambda}, \qquad \widetilde{\mathcal{WH}}^{-}_{i,n} = \bigoplus_{\lambda \neq \lambda_\circ} \widetilde{\mathcal{WH}}_{\lambda},$$

and where both sums are over partitions $\lambda \vdash n$ into i blocks.

Lemma 6.3.12. *The submodules* $\widetilde{\mathcal{WH}}^+_{i,n}$ *vanish in the range* $n < 2i$.

Proof. This follows from the simple combinatorial observation that if $n < 2i$ then any partition $\lambda \vdash n$ of length i must contain singletons. Therefore, in this range, there are no partitions λ satisfying $\lambda = \lambda_\circ$. □

Recall the definition of restriction of a PD-module to a fixed rank $r \in \mathbb{N}$ given in Definition 5.3.4. Applied to the PD-module $\widetilde{\mathcal{WH}}_\bullet$ this reads as follows.

$$\widetilde{\mathcal{WH}}^{(r)}_{i,n} = \begin{cases} \widetilde{\mathcal{WH}}_{i,n} & i = n - r \\ 0 & \text{else} \end{cases}$$

Theorem 6.3.13. *The* PD*-module* $\widetilde{\mathcal{WH}}$ *is finitely generated in rank. Moreover, the generators lie in the* PD*-submodule* $\widetilde{\mathcal{WH}}^+$.

Proof. Fix a rank $r \in \mathbb{N}$. We show there is a surjection,

$$\bigoplus_{j=1}^{r} N(j, r+j) \twoheadrightarrow \widetilde{\mathcal{WH}}^{(r)},$$

sending $f \in N(j, r+j)$ to $f_*(\widetilde{\mathcal{WH}}^+_{j,r+j})$. By Lemma 6.3.12 we see that the sum ranges over the support of $(\widetilde{\mathcal{WH}}^+)^{(r)}$. The result now follows from Proposition 6.3.11 upon noting that $\widetilde{\mathcal{WH}}^+$ contains all the modules $\widetilde{\mathcal{WH}}_{\lambda_\circ}$. □

We visualize the PD-module $\widetilde{\mathcal{WH}}_\bullet$ in Fig. 6.1 where we highlight its generators.

The PD-module $\widetilde{\mathcal{WH}}$ satisfies representation stability in the sense of Definition 5.5.2. Specifically, we have the following result.

Corollary 6.3.14. *Fix a rank* $r \in \mathbb{N}$ *and partitions* λ, μ. *Then the multiplicity* $c_{\lambda\mu}(n)$ *of,*

$$P(\lambda)_{n-r} \boxtimes P(\mu)_n,$$

in $\widetilde{\mathcal{WH}}_{n-r,n}$ *stabilizes. Concretely, there exist constants* N, C *such that,*

$$c_{\lambda\mu}(n) = C,$$

for all $n \geq N$*.*

6.3.4 Relation to configuration spaces on $\mathbb{R}^d$

Let $\mathrm{Conf}_n(X)$ denote the n-point configuration space,

$$\mathrm{Conf}_n(X) = \{(x_1, \ldots, x_n) \in X^n : x_i \neq x_j, 1 \leq i < j \leq n\}.$$

The S_n-action permuting the coordinates on $\mathrm{Conf}_n(X)$ induces an action on the (rational) cohomology $H^*(\mathrm{Conf}_n(X))$. These spaces have been extensively studied, including in the context of the FI-modules. See [5] for more details, including the celebrated result of Church that, for X a connected, orientable manifold, and for fixed $i \geq 1$, the sequence of S_n-modules,

$$\{H^i(\mathrm{Conf}_n(X))\}_n,$$

satisfies representation stability.

In [18], Hersh-Reiner analyzed the case $X = \mathbb{R}^d$ where they improved the bounds on the representation stability of Church. Central to that result is the connection between the cohomology of the configuration spaces $\mathrm{Conf}_n(\mathbb{R}^d)$ and the Whitney homology $\mathcal{WH}_{i,n}$, and in particular, the observation (Corollary 2.10, [18]) that the S_n-modules $\mathcal{WH}_{i,n}$ form a single finitely generated FI-module. We are able to recover that result as a corollary of Theorem 6.3.13.

Proposition 6.3.15. *Fix a rank* $r \in \mathbb{N}$*. There is a finitely generated* FI*-module* $V^{(r)}$ *satisfying,*

$$V_n^{(r)} = \mathcal{WH}_{n-r,n}.$$

Proof. Let $V^{(r)} := \mathcal{I}(\widetilde{\mathcal{WH}}^{(r)})$. Since $\widetilde{\mathcal{WH}}$ is finitely generated in rank, we have, by Lemma 5.5.5, that the FI-module $V^{(r)}$ is finitely generated. By Lemma 5.5.5 we have,

$$V_n^{(r)} = \mathcal{I}(\widetilde{\mathcal{WH}}^{(r)})_n \cong \left(\widetilde{\mathcal{WH}}_{n-r,n}\right)^{\mathrm{S}_i},$$

and the result now follows from Lemma 6.2.15. □

Remark 6.3.16. It is not hard to see that, in the notation of Theorem 6.2.19, the coefficients $c_{\lambda\mu}$ computed by Algorithm 4 can equivalently be described as,

$$P_\mu \otimes P_\mu \left[\!\left[\mathsf{Lie}\right]\!\right] \cong \bigoplus_\lambda c_{\lambda\mu} P_\mu \otimes P_\lambda.$$

It follows that with only a slight modification to the algorithm we can compute the irreducible decomposition of the extended Whitney homology of the lattice of set partitions. Concretely, let $d_{\lambda\mu}$ denote the multiplicity of P_λ in,

$$P_\mu^\vee \left[\!\left[\widehat{\mathsf{Lie}}\right]\!\right].$$

We should thus modify Algorithm 4 as follows:

1. Replace the Lie pieces with twisted Lie pieces, that is, the collection of all irreducible partitions appearing in $\widehat{\mathsf{Lie}}$.
2. Modify the definition of an assembly map (Definition 4.2.14) as follows. Let $\underline{\mu} := (\mu_1, \ldots, \mu_k)$ then redefine assembly as,

$$\underline{\mu}^\vee \smile (l_{i_1}, \ldots, l_{i_k}).$$

With these minor changes we are able to repurpose Algorithm 4 to compute the structure constants $d_{\lambda\mu}$, which in turn completely describe the irreducible bimodule structure of the extended Whitney homology. In Fig. 6.2 is a visualization of these structure constants for $\lambda, \mu < 11$.

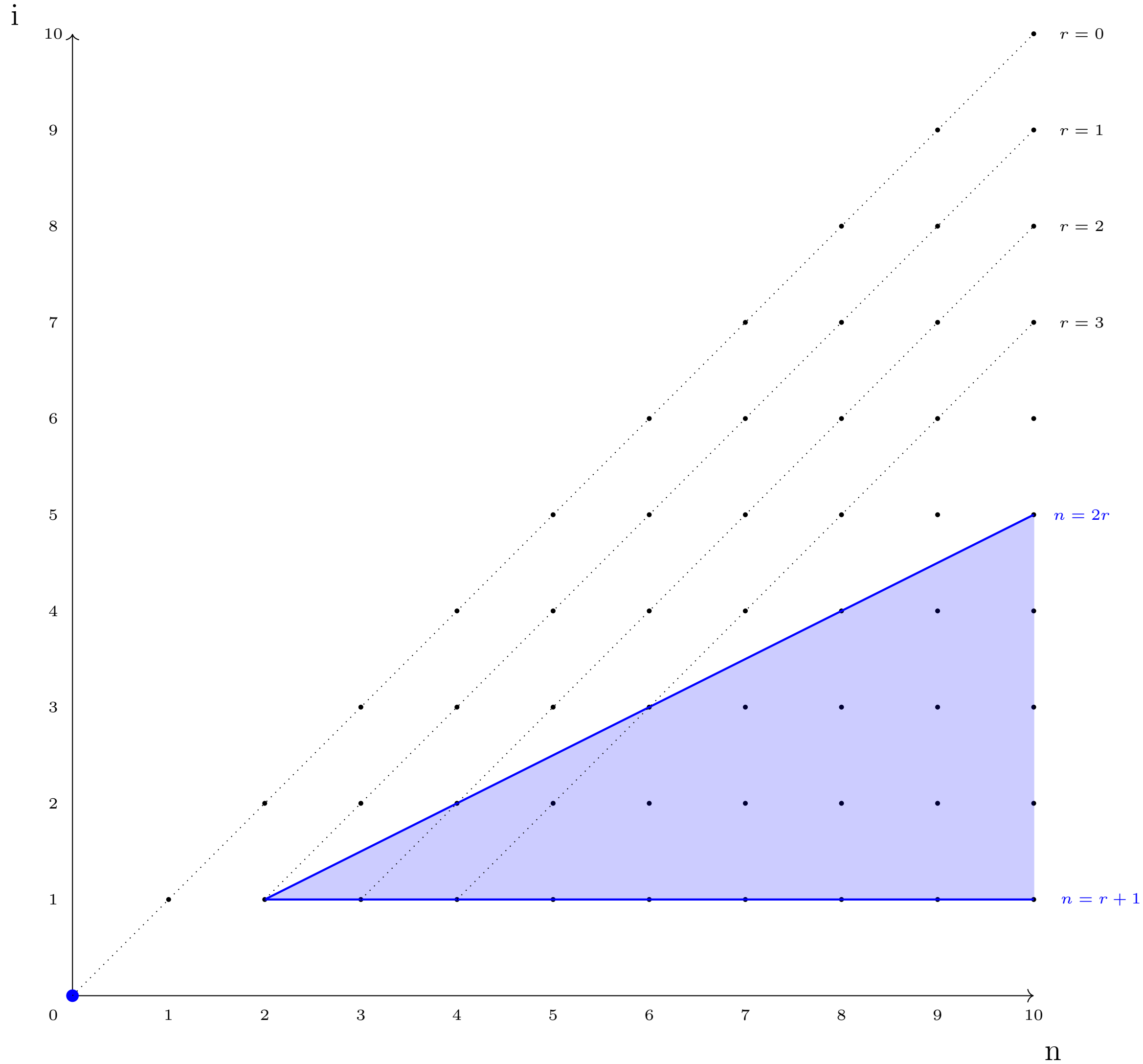


Figure 6.1: A schematic for the PD-modules $\widetilde{\mathcal{WH}}_\bullet$. The coordinate (i, n) represents the module $\widetilde{\mathcal{WH}}_{i,n}$. The generators for $\widetilde{\mathcal{WH}}_{i,n}$ live in the blue triangular region along the dashed line of the corresponding rank.

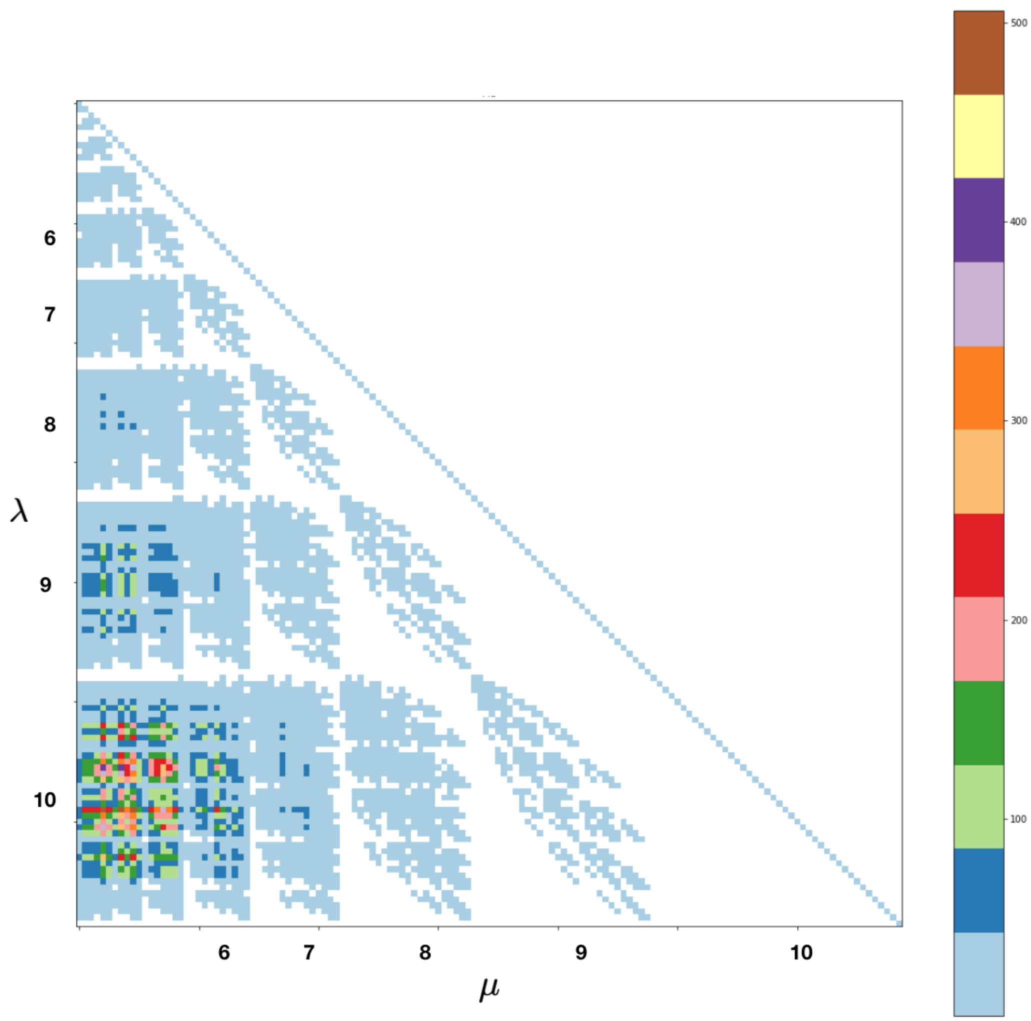


Figure 6.2: The structure coefficients $d_{\lambda\mu}$, for partitions $|\lambda|, |\mu| < 11$, associated to the extended Whitney homology. These coefficients were computed by Algorithm 4.

BIBLIOGRAPHY

[1] Aguiar, M., and Mahajan, S. Monoidal Functors, Species and Hopf Algebras. *CRM Monograph Series Vol 29* (2010).

[2] Aguiar, M., and Mahajan, S. *Topics in hyperplane arrangements*, vol. 226 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2017.

[3] Bestvina, M., and Feighn, M. The topology at infinity of $\mathrm{Out}(F_n)$. *Invent. Math. 140*, 3 (2000), 651–692.

[4] Brown, K. *Cohomology of Groups.* Springer-Verlag, New York, NY, 1982.

[5] Church, T. Homological stability for configuration spaces of manifolds. *Invent. Math. 188*, 2 (2012), 465–504.

[6] Church, T., and Ellenberg, J. S. Homology of FI-modules. *Geom. Topol. 21*, 4 (2017), 2373–2418.

[7] Church, T., Ellenberg, J. S., and Farb, B. FI-modules and stability for representations of symmetric groups. *Duke Math. J. 164*, 9 (2015), 1669–1732.

[8] Church, T., Ellenberg, J. S., Farb, B., and Nagpal, R. FI-modules over Noetherian rings. *Geom. Topol. 18*, no. 5 (2014), 2951–2984.

[9] Church, T., and Farb, B. Representation theory and homological stability. *Adv. in Math. 245* (2013), 250–314.

[10] Conant, J., Hatcher, A., Kassabov, M., and Vogtmann, K. Assembling homology classes in automorphism groups of free groups. arXiv:1501.02351v2. *Comment. Math. Helv. 91*, 4 (2016), 751–806.

[11] Conant, J., and Kassabov, M. Hopf algebras and invariants of the Johnson cokernel. *Algebr. Geom. Topol. 16*, no. 4 (2016), 2325–2363.

[12] Etingov, P., Golberg, O., Hensel, S., Liu, T., Schwendner, A., Vaintrob, D., and Yudovina, E. *Introduction to representation theory*, vol. 59. American Mathematical Society, Providence, RI, 2011.

[13] Fulton, W., and Harris, J. *Representation Theory. A First Course*. Graduate Texts in Mathematics, 129. Readings in Mathematics. Springer-Verlag, New York, 1991.

[14] Garsia, A. M., and Goupil, A. Character polynomials, their q-analogs, and the Kronecker product. *Electron. J. Combin. 16*, no. 2 (2009).

[15] Hatcher, A. Homological stability for automorphism groups of free groups. *Comment. Math. Helvetici*, 70 (1995), 39–62.

[16] Hatcher, A., and Vogtmann, K. Rational homology of $\mathrm{Aut}(F_n)$. *Math. Res. Let. 5* (1998), 759–780.

[17] Hatcher, A., and Vogtmann, K. Homological stability for outer automorphism groups of free groups. *Algebraic and Geometric Topology 4* (2004), 1253–1272. See also Erratum (with Nathalie Wahl) in v.6 (2006), 573–579.

[18] Hersh, P., and Reiner, V. Representation stability for cohomology of configuration spaces in $\mathbb{R}^d$. *Int. Math. Res. Not. IMRN*, 5 (2017), 1422–1486.

[19] James, G., and Liebeck, M. *Representations and characters of groups*, Second ed. Cambridge University Press, 2001.

[20] Klyachko, A. A. e. Lie elements in a tensor algebra. *Sibirsk. Mat. Ž. 15* (1974), 1296–1304, 1430.

[21] Lehrer, G. I., and Solomon, L. On the action of the symmetric group on the cohomology of the complement of its reflecting hyperplanes. *J. Algebra*, 104 (1986), 410–424.

[22] Loday, J.-L., and Vallette, B. *Algebraic operads*, vol. 346 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2012.

[23] Loehr, N., and Warrington, S. Quasisymmetric expansions of Schur-function plethysms. *Proc. Amer. Math. Soc. 140*, 4 (2012), 1159–1171.

[24] MacLane, S. *Categories for the working mathematician*, vol. Vol. 5. Springer-Verlag, New York-Berlin, 1971.

[25] Ramos, E. Generalized representation stability and FI_d-modules. *Proc. Amer. Math. Soc. 145*, 11 (2017), 4647–4660.

[26] Riehl, E. *Categorical homotopy theory*, vol. 24 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2014.

[27] Rolland, R. J. On the cohomology of pure mapping class groups as FI-modules. *Journal of Homotopy and Related Structures 10*, 3 (2015), 401–424.

[28] Sam, S., and Snowden, A. Stability patterns in representation theory. *Forum Math. Sigma 3 e11*, 108pp ((2015)).

[29] Sam, S., and Snowden, A. Gröbner methods for representations of combinatorial categories. *J. Amer. Math. Soc. 30*, no. 1 ((2017)), 159–203.

[30] Stanley, R. P. Some aspects of groups acting on finite posets. *J. Combinatorial Theory Ser. A 32* (1982), 132–161.

[31] Stanley, R. P. *Enumerative Combinatorics*, vol. 2. Cambridge University Press, 1999.

[32] Sundaram, S. The homology representations of the symmetric group on Cohen-Macaulay subposets of the partition lattice. *Adv. Math.*, 104 (1994), 225–296.

[33] Weibel, C. *An introduction to homological algebra.* Cambridge Strudies in Advances Mathematics, Cambridge University Press, Cambridge, 1994.